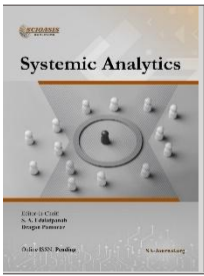


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# On Min-Max Goal Programming Approach for Solving Piecewise Quadratic Fuzzy Multi-Objective De Novo Programming Problems

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## Abstract

De novo programming is considered an essential tool for establishing optimal system design. This paper studies the Multi-Objective De Novo Programming (MODNP) problem with Piecewise Quadratic Fuzzy (PQF) data in the objective function coefficients. One of the best interval approximations, namely, the close interval approximation of the PQF number, is applied to solve the MODNP problem. A necessary and sufficient condition for the solution from the efficiency standpoint is established. A Min-max goal programming approach with positive and negative ideals is proposed to obtain optimal compromise system design. The stability set of the first kind corresponding to the optimal system design is defined and determined. The stability set of the first kind corresponding to the optimal system design is determined. The steps of the proposed solution approach are illustrated through numerical examples.

**Keywords:** Multi-objective de novo programming, Piecewise quadratic fuzzy numbers, Close interval approximation, α-Fuzzy efficient, Goal programming, Optimal system design, Parametric study.

## 1 | Introduction

Multi-objective decision-making solution approaches have an essential role in decision problem solutions. According to the Decision Maker (DM) influence in the optimization process, Multi-Objective Optimization (MO) methods can be classified as [1]:

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I. Methods where DM does not provide information (no-preference methods).

II. Methods where a posteriori information is used (posterior methods).

III. Methods where a priori information is used (priori methods).

IV. Methods where progressive information is used (interactive methods).

Wang and Chaing [2] applied the user preference-enabling method to solve general constrained nonlinear MO problems. Kundu and Islam [3] introduced an interactive method to design a highly reliable and productive system with minimum cost to solve multi-objective reliability optimization problems. Waliv et al. [4] studied the effect of capital investment and warehouse space on profits and shortage cost through sensitivity analysis and compared the efficiency of fuzzy nonlinear programming and intuitionistic fuzzy optimization techniques to obtain the solution. Ahmed [5] proposed a method to solve MO problems with intuitionistic fuzzy parameters. Liu et al. [6] introduced a new systematic method for determining an optimal operation scheme for minimizing octane number loss and operational risks.

De Novo Programming (DNP) deals with the design of an optimal system design. Many researchers have studied the DNP problem (for instance, [7]–[13].

Fuzzy sets theory, introduced by Zadeh [14], makes this possible. Fuzzy numerical data can be represented by employing fuzzy subsets of the real line, known as fuzzy numbers. Dubois [15] extended the use of algebraic operations on real numbers to fuzzy numbers by use of a fuzzification principle. Despite vast decision-making experience, the decision-maker cannot consistently articulate the goals precisely. Decision-making in a fuzzy environment, developed by Bellman and Zadeh [16], improved and was a great help in managing decision problems. Zimmermann [17] proposed the fuzzy set theory and its applications. Many approaches have been introduced for dealing with DNP problems [18], [19].

Goal Programming (GP) is one of the critical approaches in the multi-objective decision-making process, which is the extension of linear programming with the achievement of target objective values. Charnes and Cooper [20] first used the GP. Many authors applied GP in their research [20]–[23].

In his earlier work, Osman [25] introduced the notions of the solvability set, the stability set of the first kind, and the stability set of the second kind and analyzed these concepts for parametric convex nonlinear programming problems. Osman and El-Banna [26] studied the stability of multi-objective nonlinear programming problems with fuzzy parameters.

This paper introduces a Piecewise Quadratic Fuzzy Multi-Objective De Novo Programming (PQF-MODNP) problem with PQF data in the objective function coefficients. A GP approach is applied to obtain the optimal system design.

The remainder of the paper is organized as follows: Section 2 introduces some preliminaries needed in this paper. Section 3 formulates the mathematical model for a PQF MODNP problem. Section 4 characterizes the $\alpha-$ efficient solutions for *Problem (2).* Section 5 investigates the goal-programming approach for obtaining optimal system design. Section 5 gives a numerical example for illustration. Finally, some concluding remarks are reported in Section 6.

## 2 | Preliminaries

To discuss the problem quickly, it recalls basic rules and findings related to fuzzy numbers, PQF numbers, close interval approximation, and its arithmetic operations.

**Definition 1 ([14]).** Fuzzy number: A fuzzy number $\widetilde{A}$ is a fuzzy set with a membership function defined as $\pi_{\widetilde{A}}(x):\ \Re \rightarrow [0,1]$, and satisfies:

I. $\widetilde{A}$ is fuzzy convex, i.e., $\pi_{\widetilde{A}}(\delta\, x + (1-\delta)\, y) \geq \min\{\pi_{\widetilde{A}}(x), \pi_{\widetilde{A}}(y)\};$ for all $x, y \in \Re\,; 0 \leq \delta \leq 1$.

II. $\widetilde{A}$ is normal, i.e., $\exists\, x_0 \in \Re$ for which $\pi_{\widetilde{A}}(x_0) = 1$.

III. $\text{Supp}\,(\widetilde{A}) = \{x \in \Re : \pi_{\widetilde{A}}(x) > 0\}$ is the support of $\widetilde{A}$.

IV. $\pi_{\widetilde{A}}(x)$ is an upper semi-continuous (i. e., for each $\alpha \in (0,1)$, the $\alpha -$ cut set $\widetilde{A}_\alpha = \{x \in \Re : \pi_{\widetilde{A}} \geq \alpha\}$ is closed.

**Definition 2 ([27]).** A Piecewise Quadratic Fuzzy Number (PQFN) is denoted by $\widetilde{A}_{PQ} = (a_1, a_2, a_3, a_4, a_5)$, where $a_1 \leq a_2 \leq a_3 \leq a_4 \leq a_5$ are real numbers and are defined by whether their membership function $\mu_{\tilde{a}_{PQ}}$ is given by (see, *Fig. 1*).

$$\mu_{\widetilde{A}_{PQ}} = \begin{cases} 0, \quad x < a_1, \\ \frac{1}{2}\frac{1}{(a_2 - a_1)^2}(x - a_1)^2, \quad a_1 \leq x \leq a_2, \\ \frac{1}{2}\frac{1}{(a_3 - a_2)^2}(x - a_3)^2 + 1, a_2 \leq x \leq a_3, \\ \frac{1}{2}\frac{1}{(a_4 - a_3)^2}(x - a_3)^2 + 1, \quad a_3 \leq x \leq a_4, \\ \frac{1}{2}\frac{1}{(a_5 - a_4)^2}(x - a_5)^2, a_4 \leq x \leq a_5, \\ 0, \quad x > a_5. \end{cases}$$

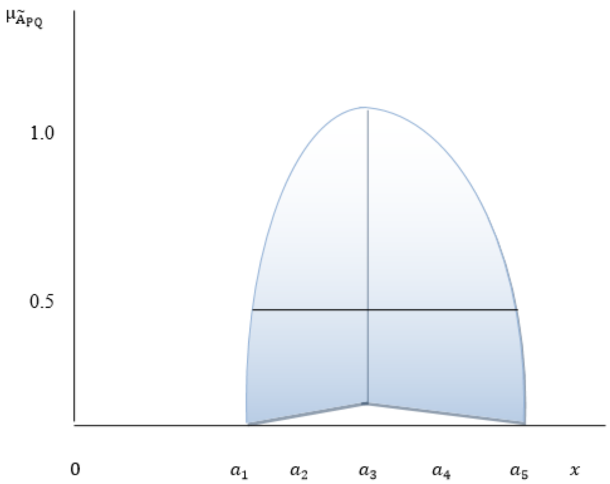


**Fig. 1. Graphical Representation of a PQFN.**

**Definition 3 ([27]).** Let $\widetilde{A}_{PQ} = (a_1, a_2, a_3, a_4, a_5)$ and $\widetilde{B}_{PQ} = (b_1, b_2, b_3, b_4, b_5)$ be two PQFNs. The arithmetic operations on $\widetilde{A}_{PQ}$ and $\widetilde{B}_{PQ}$ are.

I. Addition: $\widetilde{A}_{PQ}(+)\widetilde{B}_{PQ} = (a_1 + b_1, a_2 + b_2, a_3 + b_3, a_4 + b_4, a_5 + b_5)$.

II. Subtraction: $\widetilde{A}_{PQ}(-)\widetilde{B}_{PQ} = (a_1 + b_5, a_2 + b_4, a_3 + b_3, a_4 + b_2, a_5 + b_1)$.

III. Scalar multiplication: $k\widetilde{A}_{PQ} = \begin{cases} (k\,a_1, k\,a_2, k\,a_3, k\,a_4, k\,a_5), k > 0, \\ (k\,a_5, k\,a_4, k\,a_3, k\,a_2, k\,a_1), k < 0. \end{cases}$

**Definition 4 ([27]).** An interval approximation $[A] = [a_\alpha^-, a_\alpha^+]$ of a PQFN $\widetilde{A}$ is called closed interval approximation if

$a_\alpha^- = \inf\{x \in \mathbb{R} : \mu_{\widetilde{A}} \geq 0.5\}$, and $a_\alpha^+ = \sup\{x \in \mathbb{R} : \mu_{\widetilde{A}} \geq 0.5\}$.

**Definition 5 ([27]).** Let $[A] = [a_\alpha^-, a_\alpha^+]$, and $[B] = [b_\alpha^-, b_\alpha^+]$ be two interval approximations of PQFN. Then the arithmetic operations are.

I. Addition: $[A](+)[B] = [a_\alpha^- + b_\alpha^-, a_\alpha^+ + b_\alpha^+]$,

II. Subtraction: $[A](-)[B] = [a_\alpha^- - b_\alpha^+, a_\alpha^+ - b_\alpha^-]$,

III. Scalar multiplication: $\alpha\,[A] = \begin{cases}[\alpha\, a_\alpha^-,\ \alpha\, a_\alpha^+], \alpha > 0,\\ [\ \alpha\, a_\alpha^+, \alpha\, a_\alpha^-], \alpha < 0.\end{cases}$

IV. Multiplication: $[A](\times)[B]$

$$\left[\frac{a_\alpha^+ b_\alpha^- + a_\alpha^- b_\alpha^+}{2}, \frac{a_\alpha^- b_\alpha^- + a_\alpha^+ b_\alpha^+}{2}\right],$$

V. Division: $[A](\div)[B]$

$$\begin{cases}\left[2\left(\frac{a_\alpha^-}{b_\alpha^- + b_\alpha^+}\right), 2\left(\frac{a_\alpha^+}{b_\alpha^- + b_\alpha^+}\right)\right], [B] > 0, b_\alpha^- + b_\alpha^+ \neq 0,\\ \left[2\left(\frac{a_\alpha^+}{b_\alpha^- + b_\alpha^+}\right), 2\left(\frac{a_\alpha^-}{b_\alpha^- + b_\alpha^+}\right)\right], [B] < 0, b_\alpha^- + b_\alpha^+ \neq 0.\end{cases}$$

VI. The order relations:

I. $[A](\lesssim)[B]$ if $a_\alpha^- \leq b_\alpha^-$ and $a_\alpha^+ \leq b_\alpha^+$ or $a_\alpha^- + a_\alpha^+ \leq b_\alpha^- + b_\alpha^+$..

II. $[A]$ is preferred to $[B]$ if and only if $a_\alpha^- \geq b_\alpha^-, a_\alpha^+ \geq b_\alpha^+$..

It is noted that $P(\mathbb{R}) \subset F(\mathbb{R})$, where $F(\mathbb{R})$, and $P(\mathbb{R})$ are the sets of all PQFNs and close in interval approximation of PQFN, respectively.

## 3 | Problem Formulation

Consider a MODNP problem with PQF objective functions coefficients as

$$\max \tilde{Z}^p(x, \tilde{c}_{pj}) = \sum_{j=1}^{n} \tilde{c}_{pj}\, x_j, p = \overline{1, P},$$

$$\min \widetilde{W}^q(x, \tilde{c}_{qj}) = \sum_{j=1}^{n} \tilde{c}_{qj} x_j, q = \overline{1, Q}.$$

Subject to **(1)**

$$x \in X = \left\{x^n \in \mathfrak{R}: \sum_{j=1}^{n} a_{ij} x_j - b_i \leq 0; \sum_{i=1}^{m} r_i x_i = B; x_j \geq 0, j = \overline{1, n}\right\}.$$

Where, $\tilde{c}_{pj}(p = \overline{1, P})$ and $\tilde{c}_{qj}(q = \overline{1, Q})$ are PQF variables on $\mathfrak{R}$ which are characterized by PQFNs., $x_j(j = \overline{1, n})$ and $x_i$ are decision variables for projects and resources, respectively. $r_i$, represents the price of resource $i$, and $B$ is the total availability budget.

It is noted that *Problem (1)* can be formulated as a continuous "knapsack" problem using the unit price of resource constraints

$$\max \tilde{Z}^p(x, \tilde{c}_{pj}) = \sum_{j=1}^{n} \tilde{c}_{pj}\, x_j, p = \overline{1, P},$$

$$\min \widetilde{W}^q(x, \tilde{c}_{qj}) = \sum_{j=1}^{n} \tilde{c}_{qj} x_j, q = \overline{1, Q}.$$

Subject to **(2)**

$$x \in X = \{x \in \mathfrak{R}^n: Vx \leq B;\ x_j \geq 0, j = \overline{1, n}\}.$$

Where, $Z^p = (Z^1, \ldots, Z^P) \in \mathfrak{R}^P, W^q = (W^1, \ldots, W^Q) \in \mathfrak{R}^Q$ , $V = (V_1, \ldots, V_n) = pA \in \mathfrak{R}^n$, $B$ is the total given budget.

**Definition 6 ([28]).** $x^* \in X$ is an $\alpha -$ efficient solution for *Problem (2)* if there is no $x \in X$ such that:

$$\mu\begin{pmatrix} Z^1(x,\tilde{c}_{1j}) \geq Z^1(x^*,\tilde{c}_{1j}),\dots,Z^{p-1}(x,\tilde{c}_{p-1,j}) \geq Z^{p-1}(x^*,\tilde{c}_{p-1,j}), \\ Z^p(x,\tilde{c}_{pj}) \geq Z^p(x^*,\tilde{c}_{pj}),Z^{p+1}(x,\tilde{c}_{p+1,j}) \geq Z^{p+1}(x^*,\tilde{c}_{p+1,j}),\dots, \\ Z^P(x,\tilde{c}_{Pj}) \geq Z^P(x^*,\tilde{c}_{Pj}); W^1(x,\tilde{c}_{1j}) \leq W^1(x^*,\tilde{c}_{1j}),\dots, \\ W^{q-1}(x,\tilde{c}_{q-1,j}) \leq W^{q-1}(x^*,\tilde{c}_{q-1,j}),W^q(x,\tilde{c}_{qj}) \leq W^q(x^*,\tilde{c}_{qj}), \\ W^{q+1}(x,\tilde{c}_{q+1,j}) \leq W^{q+1}(x^*,\tilde{c}_{q+1,j}),\dots,W^Q(x,\tilde{c}_{Qj}) \leq W^Q(x^*,\tilde{c}_{Qj}) \end{pmatrix} \geq \alpha. \tag{3}$$

On the account of the extension principle,

$$\mu\begin{pmatrix} Z^1(x,\tilde{c}_{1j}) \geq Z^1(x^*,\tilde{c}_{1j}),\dots,Z^{p-1}(x,\tilde{c}_{p-1,j}) \geq Z^{p-1}(x^*,\tilde{c}_{p-1,j}), \\ Z^p(x,\tilde{c}_{pj}) \geq Z^p(x^*,\tilde{c}_{pj}),Z^{p+1}(x,\tilde{c}_{p+1,j}) \geq Z^{p+1}(x^*,\tilde{c}_{p+1,j}),\dots, \\ Z^P(x,\tilde{c}_{Pj}) \geq Z^P(x^*,\tilde{c}_{Pj}); W^1(x,\tilde{c}_{1j}) \leq W^1(x^*,\tilde{c}_{1j}),\dots, \\ W^{q-1}(x,\tilde{c}_{q-1,j}) \leq W^{q-1}(x^*,\tilde{c}_{q-1,j}),W^q(x,\tilde{c}_{qj}) \leq W^q(x^*,\tilde{c}_{qj}), \\ W^{q+1}(x,\tilde{c}_{q+1,j}) \leq W^{q+1}(x^*,\tilde{c}_{q+1,j}),\dots,W^Q(x,\tilde{c}_{Qj}) \leq W^Q(x^*,\tilde{c}_{Qj}), \end{pmatrix} = \tag{4}$$

$$\underset{(c_1,\dots,c_P;\,c_1,\dots,c_Q)\in C\times D}{\text{Sup}} \min\begin{pmatrix} \mu_{\tilde{c}_1}(c_1),\dots,\mu_{\tilde{c}_{p-1}}(c_{p-1}),\mu_{\tilde{c}_p}(c_p),\mu_{\tilde{c}_{p+1}}(c_{p+1}),\dots,\mu_{\tilde{c}_P}(c_P); \\ \mu_{\tilde{c}_1}(c_1),\dots,\mu_{\tilde{c}_{q-1}}(c_{q-1}),\mu_{\tilde{c}_q}(c_q),\mu_{\tilde{c}_{q+1}}(c_{q+1}),\dots,\mu_{\tilde{c}_Q}(c_Q) \end{pmatrix}.$$

Where

$$C = \begin{Bmatrix} (c_1,\dots,c_P) \in \Re^{P(n)}: Z^1(x,c_{1j}) \geq Z^1(x^*,c_{1j}),\dots, \\ Z^{p-1}(x,c_{p-1,j}) \geq Z^{p-1}(x^*,c_{p-1,j}),Z^p(x,c_{pj}) \geq Z^p(x^*,c_{pj}), \\ Z^{p+1}(x,c_{p+1,j}) \geq Z^{p+1}(x^*,c_{p+1,j}),\dots,Z^P(x,c_{Pj}) \geq Z^P(x^*,c_{Pj}) \end{Bmatrix}, \tag{5}$$

$$D = \begin{Bmatrix} (c_1,\dots,c_Q) \in \Re^{P(n)}: W^1(x,c_{1j}) \leq W^1(x^*,c_{1j}),\dots, \\ W^{q-1}(x,c_{q-1,j}) \leq W^{q-1}(x^*,c_{q-1,j}),W^q(x,c_{qj}) \leq W^q(x^*,c_{qj}), \\ W^{q+1}(x,c_{q+1,j}) \leq W^{q+1}(x^*,c_{q+1,j}),\dots,W^Q(x,c_{Qj}) \leq W^Q(x^*,c_{Qj}) \end{Bmatrix}.$$

$\mu_{\tilde{c}_p}(p=\overline{1,P})$ and $\mu_{\tilde{c}_q}(q=\overline{1,Q})$ are $(n\times 1)$- ary $\alpha -$level sets.

## 4 | Characterizing of $\alpha$ −Efficient Solutions for Problem (2)

To characterize the $\alpha -$ efficient solutions for *Problem (2)*, let us consider the following $\alpha -$ parametric MODNP problem:

$$\max Z^p(x,c_{pj}) = \sum_{j=1}^{n} c_{pj}\,x_j\,, p=\overline{1,P},$$

$$\min W^q(x,c_{qj}) = \sum_{j=1}^{n} c_{qj}x_j, q=\overline{1,Q}.$$

Subject to (6)

$x \in X$, $c_{pj} \in (\tilde{c}_{pj})_\alpha$ and $c_{qj} \in (\tilde{c}_{qj})_\alpha$.

Where, $(\tilde{c}_{pj})$ and $(\tilde{c}_{qj})$ denote the $\alpha -$ level sets of the fuzzy variables $\tilde{c}_{pj}$ and $\tilde{c}_{qj}$. Based on the assumptions of the convexity,$\mu_{\tilde{c}_{pj}}, (\tilde{c}_{pj})_\alpha$; $\mu_{\tilde{c}_{qj}}, (\tilde{c}_{qj})_\alpha$, $(j=\overline{1,n}; p=\overline{1,P}; q=\overline{1,Q})$ are close intervals approximations of real numbers that are denoted by $[c^-_{pj}(\alpha), c^+_{pj}(\alpha)]$ and $[c^-_{qj}(\alpha), c^+_{qj}(\alpha)]$. Let $\chi^\alpha_{pj}$ and $\chi^\alpha_{qj}$ be the sets of $n\times 1$

matrices$\left(c_{pj}\right)$ with $\left[c_{pj}^{-}(\alpha),\ c_{pj}^{+}(\alpha)\right]$ $(, p=\overline{1,P})$ and $\left(c_{qj}\right)$ with $\left[c_{qj}^{-}(\alpha),\ c_{qj}^{+}(\alpha)\right], (q=\overline{1,Q})$. *Problem (6)* can be rewritten as

$$\max Z^{p}\left(x, c_{pj}\right), p=\overline{1,P},$$
$$\min W^{q}\left(x, c_{qj}\right), q=\overline{1,Q}.$$
Subject to **(7)**
$$x \in X,\ c_{pj} \in \chi_{pj}^{\alpha} \text{ and } c_{qj} \in \chi_{qj}^{\alpha}, p=\overline{1,P}; q=\overline{1,Q}.$$

**Theorem 1.** $x^{*} \in X$ is said to be an $\alpha-$efficient solution for *Problem (2)* if and only if $x^{*} \in X$ is an $\alpha-$parametric efficient solution for *Problem (6)*.

Proof: Necessity: Let $x^{*} \in X$ be an $\alpha-$efficient solution for *Problem (2)* and $x^{*} \in X$ be not a $\alpha-$n parametric efficient solution for *Problem (6)*. Then there are $x^{1} \in X, s_{pj} \in \chi_{pj}^{\alpha}$ and $t_{qj} \in \chi_{qj}^{\alpha}, p=\overline{1,P}; q=\overline{1,Q}; j=\overline{1,n}.$ such that

$$Z^{p}\left(x, s_{pj}\right) \geq Z^{p}\left(x^{*}, s_{pj}\right) \text{ and } W^{q}\left(x, t_{qj}\right) \leq W^{q}\left(x^{*}, t_{qj}\right).$$

Since, $s_{pj} \in \chi_{pj}^{\alpha}$ and $t_{qj} \in \chi_{qj}^{\alpha}, p=\overline{1,P}; q=\overline{1,Q}; j=\overline{1,n}.$ we have

$$\mu\left(\begin{array}{c} Z^{1}\left(x^{1}, \tilde{c}_{1j}\right) \geq Z^{1}\left(x^{*}, \tilde{c}_{1j}\right), \ldots, Z^{p-1}\left(x^{1}, \tilde{c}_{p-1,j}\right) \geq Z^{p-1}\left(x^{*}, \tilde{c}_{p-1,j}\right), \\ Z^{p}\left(x^{1}, \tilde{c}_{pj}\right) \geq Z^{p}\left(x^{*}, \tilde{c}_{pj}\right), Z^{p+1}\left(x^{1}, \tilde{c}_{p+1,j}\right) \geq Z^{p+1}\left(x^{*}, \tilde{c}_{p+1,j}\right), \ldots, \\ Z^{P}\left(x^{1}, \tilde{c}_{Pj}\right) \geq Z^{P}\left(x^{*}, \tilde{c}_{Pj}\right); W^{1}\left(x^{1}, \tilde{c}_{1j}\right) \leq W^{1}\left(x^{*}, \tilde{c}_{1j}\right), \ldots, \\ W^{q-1}\left(x^{1}, \tilde{c}_{q-1,j}\right) \leq W^{q-1}\left(x^{*}, \tilde{c}_{q-1,j}\right), W^{q}\left(x^{1}, \tilde{c}_{qj}\right) \leq W^{q}\left(x^{*}, \tilde{c}_{qj}\right), \\ W^{q+1}\left(x^{1}, \tilde{c}_{q+1,j}\right) \leq W^{q+1}\left(x^{*}, \tilde{c}_{q+1,j}\right), \ldots, W^{Q}\left(x^{1}, \tilde{c}_{Qj}\right) \leq W^{Q}\left(x^{*}, \tilde{c}_{Qj}\right) \end{array}\right) \geq \alpha.$$

Contradiction the assumption that $x^{*} \in X$ be an $\alpha-$efficient solution for *Problem (2)*.

Sufficiency: Let $x^{*} \in X$ be an $\alpha-$parametric efficient solution for *Problem (6)* and $x^{*} \in X$ be not an $\alpha-$efficient solution for *Problem (2)*. Then, there are $x^{1} \in X, p=\{1, \ldots, P\}$ and $q=\{1, \ldots, Q\}$ such that

$$\mu\left(\begin{array}{c} Z^{1}\left(x^{2}, \tilde{c}_{1j}\right) \geq Z^{1}\left(x^{*}, \tilde{c}_{1j}\right), \ldots, Z^{p-1}\left(x^{2}, \tilde{c}_{p-1,j}\right) \geq Z^{p-1}\left(x^{*}, \tilde{c}_{p-1,j}\right), \\ Z^{p}\left(x^{2}, \tilde{c}_{pj}\right) \geq Z^{p}\left(x^{*}, \tilde{c}_{pj}\right), Z^{p+1}\left(x^{2}, \tilde{c}_{p+1,j}\right) \geq Z^{p+1}\left(x^{*}, \tilde{c}_{p+1,j}\right), \ldots, \\ Z^{P}\left(x^{2}, \tilde{c}_{Pj}\right) \geq Z^{P}\left(x^{*}, \tilde{c}_{Pj}\right); W^{1}\left(x^{2}, \tilde{c}_{1j}\right) \leq W^{1}\left(x^{*}, \tilde{c}_{1j}\right), \ldots, \\ W^{q-1}\left(x^{2}, \tilde{c}_{q-1,j}\right) \leq W^{q-1}\left(x^{*}, \tilde{c}_{q-1,j}\right), W^{q}\left(x^{2}, \tilde{c}_{qj}\right) \leq W^{q}\left(x^{*}, \tilde{c}_{qj}\right), \\ W^{q+1}\left(x^{2}, \tilde{c}_{q+1,j}\right) \leq W^{q+1}\left(x^{*}, \tilde{c}_{q+1,j}\right), \ldots, W^{Q}\left(x^{2}, \tilde{c}_{Qj}\right) \leq W^{Q}\left(x^{*}, \tilde{c}_{Qj}\right) \end{array}\right) \geq \alpha, \quad \textbf{(8)}$$

i.e.,

$$\sup_{(c_{1},\ldots,c_{P};\, c_{1},\ldots,c_{Q}) \in \hat{C} \times \hat{D}} \min\left(\begin{array}{c} \mu_{\tilde{c}_{1j}}\left(c_{1j}\right), \ldots, \mu_{\tilde{c}_{p-1,j}}\left(c_{p-1,j}\right), \mu_{\tilde{c}_{pj}}\left(c_{pj}\right), \mu_{\tilde{c}_{p+1,j}}\left(c_{p+1,j}\right), \ldots, \mu_{\tilde{c}_{Pj}}\left(c_{Pj}\right); \\ \mu_{\tilde{c}_{1j}}\left(c_{1j}\right), \ldots, \mu_{\tilde{c}_{q-1,j}}\left(c_{q-1,j}\right), \mu_{\tilde{c}_{qj}}\left(c_{qj}\right), \mu_{\tilde{c}_{q+1,j}}\left(c_{q+1,j}\right), \ldots, \mu_{\tilde{c}_{Qj}}\left(c_{Qj}\right) \end{array}\right) \geq \alpha.$$

Where

$$\hat{C}=\left\{\begin{array}{c} \left(c_{1}, \ldots,\ c_{P}\right) \in \Re^{P \times Q(n \times 1)}: Z^{1}\left(x, c_{1j}\right) \geq Z^{1}\left(x^{*}, c_{1j}\right), \ldots, \\ Z^{p-1}\left(x, c_{p-1,j}\right) \geq Z^{p-1}\left(x^{*}, c_{p-1,j}\right), Z^{p}\left(x, c_{pj}\right) \geq Z^{p}\left(x^{*}, c_{pj}\right), \\ Z^{p+1}\left(x, c_{p+1,j}\right) \geq Z^{p+1}\left(x^{*}, c_{p+1,j}\right), \ldots, Z^{P}\left(x, c_{Pj}\right) \geq Z^{P}\left(x^{*}, c_{Pj}\right). \end{array}\right\}.$$

And

$$\widehat{D}=\left\{\begin{matrix}(c_1,\dots,c_Q)\in\Re^{Q(n\times1)}: W^1(x^2,c_{1j})\le W^1(x^*,c_{1j}),\dots,\\ W^{q-1}(x^2,c_{q-1,j})\le W^{q-1}(x^*,c_{q-1,j}), W^q(x^2,c_{qj})\le W^q(x^*,c_{qj}),\\ W^{q+1}(x^2,c_{q+1,j})\le W^{q+1}(x^*,c_{q+1,j}),\dots,W^Q(x^2,c_{Qj})\le W^Q(x^*,c_{Qj}).\end{matrix}\right\}.$$

In the case for existing the supremum, there is $(u_{1j},\dots,u_{Pj})\in\widehat{C}$ and $(v_{1j},\dots,v_{Qj})\in\widehat{D}$ with

$\min\left\{\mu_{\tilde{c}_{1j}}(u_{1j}),\dots,\mu_{\tilde{c}_{Pj}}(u_{1j})\right\}<\alpha$, and $\min\left\{\mu_{\tilde{c}_{1j}}(v_{1j}),\dots,\mu_{\tilde{c}_{1j}}(v_{Qj})\right\}<\alpha$, then

$$\sup_{(c_1,\dots,c_P;\,c_1,\dots,c_Q)\in\widehat{C}\times\widehat{D}}\min\begin{pmatrix}\mu_{\tilde{c}_{1j}}(c_{1j}),\dots,\mu_{\tilde{c}_{p-1,j}}(c_{p-1,j}),\mu_{\tilde{c}_{pj}}(c_{pj}),\mu_{\tilde{c}_{p+1,j}}(c_{p+1,j}),\dots,\mu_{\tilde{c}_{Pj}}(c_{Pj});\\ \mu_{\tilde{c}_{1j}}(c_{1j}),\dots,\mu_{\tilde{c}_{q-1,j}}(c_{q-1,j}),\mu_{\tilde{c}_{qj}}(c_{qj}),\mu_{\tilde{c}_{q+1,j}}(c_{q+1,j}),\dots,\mu_{\tilde{c}_{Qj}}(c_{Qj})\end{pmatrix}<\alpha.$$

This is a contradiction (4). Then there are $(u_{1j},\dots,u_{Pj})\in\widehat{C}$ and $(v_{1j},\dots,v_{Qj})\in\widehat{D}$ with

$\min\left\{\mu_{\tilde{c}_{1j}}(u_{1j}),\dots,\mu_{\tilde{c}_{Pj}}(u_{Pj})\right\}\ge\alpha$, and $\min\left\{\mu_{\tilde{c}_{1j}}(v_{1j}),\dots,\mu_{\tilde{c}_{1j}}(v_{Qj})\right\}\ge\alpha$, i. e.,

$$u_{pj}\in\chi_{pj}^{\alpha}\ (p=\overline{1,P})\ \text{and}\ v_{qj}\in\chi_{qj}^{\alpha}\ (q=\overline{1,Q}). \quad (9)$$

From *Eqs. (4)* and *(10)*, we conclude that $x^*\in X$ is an $\alpha$ −parametric efficient solution for *Problem (6)*.

## 5 | Min-max GP Approach

$\alpha$ −Parametric MODNP *Problem (6)* can be demonstrated using a min-max GP approach as $\min d$

Subject to **(10)**

$$Z^p(x,c_{pj})+l_p-f_p=Z^{p*}, l_p\le d,$$
$$\gamma_p\frac{l_p}{y_p}\le d, y_p=Z^{p*}-Z^{p-}.$$
$$W^q(x,c_{qj})+l_q-f_q=W^{q*},$$
$$\delta_q\frac{d_q}{y_q}\le d, W^{q-}-W^{q*}.$$
$$Vx\le B,$$
$$c_{pj}\in[c_{pj}^-(\alpha),\ c_{pj}^+(\alpha)], c_{qj}\in[c_{qj}^-(\alpha),\ c_{qj}^+(\alpha)], p=\overline{1,P};\ q=\overline{1,Q};\ j=\overline{1,n}, 0\le d\le 1.$$

Where, $Z^{p*}=\max Z^p$, and $W^{q*}=\min W^q$, are the positive ideal solutions, respectively. Also, $Z^{p*}=\min Z^p$, and $W^{q*}=\max W^q$ are the negative ideal solutions; respectively, $\gamma_p$ and $\delta_q$ are positive weights, $y_p=Z^{p*}-Z^{p-}$, and $y_q=W^{q-}-W^{q*}$ are the normalization of the positive and negative ideal solutions, respectively.

Now, let us determine the stability set of the first kind $S(\hat{x},\hat{c})$ corresponding to the $\alpha$ −optimal solution by applying the following condition:

$$\vartheta_{pj}(\hat{c}_{pj}-c_{pj}^+)=0, p=\overline{1,P}; j=\overline{1,n},$$
$$\theta_{pj}(c_{pj}^- -\hat{c}_{pj})=0, p=\overline{1,P}; j=\overline{1,n},$$
$$\vartheta_{qj}(\hat{c}_{qj}-c_{qj}^+)=0, q=\overline{1,Q}; j=\overline{1,n},$$
$$\theta_{qj}(c_{qj}^- -\hat{c}_{qj})=0, q=\overline{1,Q}; j=\overline{1,n}.$$

Where

$[c_{pj}^-(\alpha),\ c_{pj}^+(\alpha)]=L_\alpha(\tilde{c}_{pj}), p=\overline{1,P}, j=\overline{1,n}$, and $[c_{qj}^-(\alpha),\ c_{qj}^+(\alpha)], q=\overline{1,Q}$.

Consider the following cases:

I. $\vartheta_{pj}>0, p\in J_1\subset\{1,\dots,P\};\ \vartheta_{pj}=0, p\notin J_1,$

$\vartheta_{qj}>0, q\in J_1\subset\{1,\dots,Q\};\ \vartheta_{qj}=0, q\notin J_1,$

$\theta_{pj} > 0, i \in J_2 \subset \{1, \dots, P\}, \theta_{pj} = 0, i \notin J_2,$

$\theta_{qj} > 0, i \in J_2 \subset \{1, \dots, Q\}, \theta_{qj} = 0, i \notin J_2.$

Let $M$ be the set of all proper subsets of $\{1, \dots, P\}$ and $\{1, \dots, Q\}$. Then

$$S_{J_1,J_2}(\hat{x}, \hat{c}) = \left\{ \begin{array}{c} \{(c_{pj}^-, c_{pj}^+), (c_{qj}^-, c_{qj}^+)\} \in \mathfrak{R}^{2p\times 2q}: c_{pj}^+ = \hat{c}_{pj}, p \in J_1, c_{pj}^+ \geq \hat{c}_{pj}, p \notin J_1, \\ c_{qj}^+ = \hat{c}_{qj}, q \in J_1, c_{qj}^+ \geq \hat{c}_{qj}, q \notin J_1; c_{pj}^- = \hat{c}_{pj}, p \in J_2, c_{pj}^- \leq \hat{c}_{pj}, p \notin J_2, \\ c_{qj}^- = \hat{c}_{qj}, q \in J_2, c_{qj}^- \leq \hat{c}_{qj}, q \notin J_2. \end{array} \right\}.$$

Hence

$$S_1(\hat{x}, \hat{c}) = \bigcup_{J_1,J_2 \in M} S_{J_1,J_2}(\hat{x}, \hat{c}).$$

II. $\vartheta_{pj}, \theta_{pj};\ \vartheta_{qj}, \theta_{qj} = 0$, then

$$S_2(\hat{x}, \hat{c}) = \left\{ \begin{array}{c} \{(c_{pj}^-, c_{pj}^+), (c_{qj}^-, c_{qj}^+)\} \in \mathfrak{R}^{2p\times 2q}: c_{pj}^+ \geq \hat{c}_{pj}, p = \overline{1, P}, \\ c_{qj}^+ \geq \hat{c}_{qj}, q = \overline{1, Q};\ c_{pj}^- \leq \hat{c}_{pj}, p = \overline{1, P}, c_{qj}^- \leq \hat{c}_{qj}, q = \overline{1, Q} \end{array} \right\}.$$

III. $\vartheta_{pj}, \theta_{pj};\ \vartheta_{qj}, \theta_{qj} > 0$, then

$$S_3(\hat{x}, \hat{c}) = \left\{ \begin{array}{c} \{(c_{pj}^-, c_{pj}^+), (c_{qj}^-, c_{qj}^+)\} \in \mathfrak{R}^{2p\times 2q}: c_{pj}^+ = \hat{c}_{pj}, p = \overline{1, P}, \\ c_{qj}^+ = \hat{c}_{qj}, q = \overline{1, Q};\ c_{pj}^- = \hat{c}_{pj}, p = \overline{1, P}, c_{qj}^- = \hat{c}_{qj}, q = \overline{1, Q} \end{array} \right\}.$$

Hence

$$S(\hat{x}, \hat{c}) = \bigcup_{k=1}^{K} S_k(\hat{x}, \hat{c}).$$

## 6 | Numerical Example

Consider the following problem

I. $\max \tilde{Z}^1 = \tilde{c}_{11}x_1 + \tilde{c}_{12}x_2 + \tilde{c}_{13}x_3,$ (Profits).

II. $\max \tilde{Z}^2 = \tilde{c}_{21}x_1 + \tilde{c}_{22}x_2 + \tilde{c}_{23}x_3,$ (Quality).

III. $\max \tilde{Z}^3 = \tilde{c}_{31}x_1 + \tilde{c}_{32}x_2 + \tilde{c}_{33}x_3,$ (Workers Satisfaction).

Subject to **(11)**

$12x_1 + 17x_2 \leq 1400,$ (Milling Machine).

$3x_1 + 9x_2 + 8x_3 \leq 1000,$ (Lathe).

$10x_1 + 13x_2 + 15x_3 \leq 1750,$ (Grinder).

$6x_1 + 0x_2 + 16x_3 \leq 1325,$ (Jig Saw).

$0x_1 + 12x_2 + 7x_3 \leq 900,$ (Drill Press).

$x_1,\ x_2,\ x_3 \geq 0.$

With the price of resources $p_1 = \$\ 0.75, p_2 = \$\ 0.6, p_3 = \$0.35, p_4 = \$0.50, p_5 = \$1.15$ and $p_6 = \$0.65$, and the budget level $B = \$4658.75$,

$\tilde{c}_{11} = (20, 30, 50, 60, 80)\ \tilde{c}_{12} = (70, 85, 100, 110, 130), \tilde{c}_{13} = (13.5, 15.5, 17.5, 20.5, 22.5),$

$, \tilde{c}_{21} = (85, 90, 92, 97, 105), \tilde{c}_{22} = (60, 70, 75, 85, 100), \tilde{c}_{23} = (30, 45, 50, 60, 75),$

$\tilde{c}_{31} = (10, 18, 25, 30, 45), \tilde{c}_{32} = (80, 90, 100, 110, 115), \tilde{c}_{33} = (55, 75, 75, 80, 95).$

*Problem (11)* can be formulated according to *Problem (2)* as

$\max \tilde{Z}^1 = \tilde{c}_{11}x_1 + \tilde{c}_{12}x_2 + \tilde{c}_{13}x_3$, (Profits).
$\max \tilde{Z}^2 = \tilde{c}_{21}x_1 + \tilde{c}_{22}x_2 + \tilde{c}_{23}x_3$, (Quality).
$\max \tilde{Z}^3 = \tilde{c}_{31}x_1 + \tilde{c}_{32}x_2 + \tilde{c}_{33}x_3$, (Workers Satisfaction).
Subject to **(12)**
$23.475x_1 + 42.675x_2 + 28.7x_3 = 4558.75,$
$x_1, x_2, x_3 \geq 0.$

The positive ideal solutions to *Problem (11)* are

$\max Z^1 = 50x_1 + 100x_2 + 17.5x_3,$
$\max Z^2 = 92x_1 + 75x_2 + 50x_3,$
$\max Z^3 = 25x_1 + 100x_2 + 75x_3.$
Subject to **(13)**
$23.475x_1 + 42.675x_2 + 28.7x_3 = 4558.75,$
$x_1, x_2, x_3 \geq 0.$

**Table 1. Positive ideal solutions to problem (13).**

| Decision Variables | $Z^1$ | $Z^2$ | $Z^3$ |
|---|---|---|---|
| $x_1$ | 0 | 194.1960 | 0 |
| $x_2$ | 106.8248 | 0 | 0 |
| $x_3$ | 0 | 0 | 158.8415 |
| | $\max Z^{1*} = 10682.48$ | $\max Z^{2*} = 17866.03$ | $\max Z^{3*} = 11913.11$ |

The negative ideal solutions to *Problem (11)* are.

$\min Z^1 = 20x_1 + 130x_2 + 13.5x_3.$
$\min Z^2 = 105x_1 + 60x_2 + 75x_3.$
$\min Z^3 = 10x_1 + 115x_2 + 55x_3.$
Subject to **(14)**
$23.475x_1 + 42.675x_2 + 28.7x_3 = 4558.75,$
$x_1, x_2, x_3 \geq 0.$

**Table 2. Negative ideal solutions to problem (14).**

| Decision variables | $Z^1$ | $Z^2$ | $Z^3$ |
|---|---|---|---|
| $x_1$ | 0 | 0 | 194.1960 |
| $x_2$ | 0 | 106.8248 | 0 |
| $x_3$ | 158.8415 | 0 | 0 |
| | $\max Z^{1*} = 2144.360$ | $\max Z^{2*} = 6409.490$ | $\max Z^{3*} = 1941.960$ |

Based on the positive and negative ideal solutions, *Problem (11)*, referring to *Problem (10)*, becomes $\min d$ subject to **(15)**

$c_{11}x_1 + c_{12}x_2 + c_{13}x_3 + l_1 - f_1 = 10682.48,$
$c_{21}x_1 + c_{22}x_2 + c_{23}x_3 + l_2 - f_2 = 17866.03,$
$c_{31}x_1 + c_{32}x_2 + c_{33}x_3 + l_3 - f_3 = 11913.11,$
$\frac{l_1}{10682.48 - 2144.360} \leq d,$
$\frac{l_2}{17866.03 - 6409.490} \leq d,$
$\frac{l_3}{11913.11 - 1941.960} \leq d,$
$23.475x_1 + 42.675x_2 + 28.7x_3 = 4558.75,$
$x_1, x_2, x_3 \geq 0, 0 \leq d \leq 1,$
$c_{11} = [30, 60], c_{12} = [85, 110], c_{13} = [15.5, 20.5],$
$c_{21} = [90, 97], c_{22} = [70, 85], c_{23} = [45, 65],$
$c_{31} = [18, 30], c_{32} = [90, 110], c_{33} = [75, 80].$

**Table 3. An optimal satisfactory solution to problem (15).**

| Decision Variables | Values |
|---|---|
| $\hat{x}_1$ | 89 |
| $\hat{c}_{11}$ | 60 |
| $\hat{c}_{12}$ | 110 |
| $\hat{c}_{13}$ | 20.5 |
| $\hat{x}_2$ | 17.311 |
| $\hat{c}_{21}$ | 97 |
| $\hat{c}_{22}$ | 85 |
| $\hat{c}_{23}$ | 80 |
| $\hat{x}_3$ | 59.76241 |
| $\hat{c}_{31}$ | 30 |
| $\hat{c}_{32}$ | 110 |
| $\hat{c}_{33}$ | 65 |
| $l_1$ | 2173.183 |
| $l_1$ | 2916 |
| $l_1$ | 2537.928 |
| $f_1$ | 0 |
| $f_1$ | 0 |
| $f_1$ | 0 |
| $\max \widehat{Z^1} = 8529.34$ | |
| $\max \widehat{Z^2} = 14885.43$ | |
| $\max \widehat{Z^3} = 8458.77$ | |
| $d = 0.2545271$ | |

It is clear that the results obtained by the proposed method are less than those obtained by Umarusman [13].

Hence, $S(\hat{x}, \hat{c})$ is determined by applying the following conditions:

$$\vartheta_{1j}\left(\hat{c}_{1j} - c_{1j}^{+}\right) = 0, j = 1, 2, 3,$$
$$\vartheta_{2j}\left(\hat{c}_{1j} - c_{1j}^{+}\right) = 0, j = 1, 2, 3,$$
$$\vartheta_{3j}\left(\hat{c}_{1j} - c_{1j}^{+}\right) = 0, j = 1, 2, 3, \vartheta_{1j}, \vartheta_{2j}, \vartheta_{3j} \geq 0.$$

We have $J_1 \subseteq \{1,2,3\}$. For $J_1 = \emptyset, \vartheta_{1j} = \vartheta_{2j} = \vartheta_{3j} = 0, j = 1, 2, 3$. Then

$$S_{J_1}(\hat{x}, \hat{c}) = \begin{Bmatrix} c \in \Re^9 : c_{11} \geq 60, c_{12} \geq 110, c_{13} \geq 20.5, c_{21} \geq 97, \\ c_{22} \geq 85, c_{23} \geq 80, c_{31} \geq 30, , c_{32} \geq 110, c_{33} \geq 65 \end{Bmatrix}.$$

For $J_2 = \{1\}, \vartheta_{1j} > 0, \vartheta_{2j} = 0, \vartheta_{3j} = 0$. Then

$$S_{J_2}(\hat{x}, \hat{c}) = \begin{Bmatrix} c \in \Re^9 : c_{11} = 60, c_{12} = 110, c_{13} = 20.5, c_{21} \geq 97, \\ c_{22} \geq 85, c_{23} \geq 80, c_{31} = 30, , c_{32} = 110, c_{33} = 65 \end{Bmatrix}.$$

For $J_3 = \{2\}, \vartheta_{1j} = 0, \vartheta_{2j} > 0, \vartheta_{3j} = 0$. Then

$$S_{J_3}(\hat{x}, \hat{c}) = \begin{Bmatrix} c \in \Re^9 : c_{11} \geq 60, c_{12} \geq 110, c_{13} \geq 20.5, c_{21} = 97, \\ c_{22} = 85, c_{23} = 80, c_{31} \geq 30, , c_{32} \geq 110, c_{33} \geq 65 \end{Bmatrix}.$$

For $J_4 = \{3\}, \vartheta_{1j} = 0, \vartheta_{2j} = 0, \vartheta_{3j} > 0$. Then

$$S_{J_4}(\hat{x}, \hat{c}) = \begin{Bmatrix} c \in \Re^9 : c_{11} \geq 60, c_{12} \geq 110, c_{13} \geq 20.5, c_{21} \geq 97, \\ c_{22} \geq 85, c_{23} \geq 80, c_{31} = 30, , c_{32} = 110, c_{33} = 65 \end{Bmatrix}.$$

For $J_5 = \{1,2\}, \vartheta_{1j} > 0, \vartheta_{2j} > 0, \vartheta_{3j} = 0$. Then

$$S_5(\hat{x}, \hat{c}) = \begin{Bmatrix} c \in \Re^9 : c_{11} = 60, c_{12} = 110, c_{13} = 20.5, c_{21} = 97, \\ c_{22} = 85, c_{23} = 80, c_{31} \geq 30, , c_{32} \geq 110, c_{33} \geq 65 \end{Bmatrix},$$

For $J_6 = \{1,3\}, \vartheta_{1j} > 0, \vartheta_{2j} = 0, \vartheta_{3j} > 0$. Then

$$S_{J_6}(\hat{x}, \hat{c}) = \begin{Bmatrix} c \in \Re^9 : c_{11} \geq 60, c_{12} \geq 110, c_{13} \geq 20.5, c_{21} = 97, \\ c_{22} = 85, c_{23} = 80, c_{31} \geq 30, , c_{32} \geq 110, c_{33} \geq 65 \end{Bmatrix}.$$

For $J_7 = \{2,3\}, \vartheta_{1j} = 0, \vartheta_{2j} > 0, \vartheta_{3j} > 0$. Then

$$S_{J_7}(\hat{x}, \hat{c}) = \begin{Bmatrix} c \in \Re^9 : c_{11} \geq 60, c_{12} \geq 110, c_{13} \geq 20.5, c_{21} = 97, \\ c_{22} = 85, c_{23} = 80, c_{31} = 30, , c_{32} = 110, c_{33} = 65 \end{Bmatrix}.$$

For $J_8 = \{1,2,3\}, \vartheta_{1j} > 0, \vartheta_{2j} > 0, \vartheta_{3j} > 0$. Then

$$S_{J_8}(\hat{x},\hat{c}) = \begin{Bmatrix} c \in \Re^9: c_{11} = 60, c_{12} = 110, c_{13} = 20.5, c_{21} = 97, \\ c_{22} = 85, c_{23} = 80, c_{31} = 30, , c_{32} = 110, c_{33}65 \end{Bmatrix}.$$

Hence

$$S(\hat{x},\hat{c}) = \bigcup_{k=1}^{8} S_k(\hat{x},\hat{c}).$$

## 7 | Conclusions

The De novo hypothesis provides meta0 optimum solutions at optimal levels for single and multi-objective programming. This paper studies DNP in an uncertain environment. Close interval approximation of the PQF number is applied to solve the MODNP problem. A necessary and sufficient condition for the solution from the efficiency standpoint has been established.

A Min-max goal programming approach with positive and negative ideals has been proposed for optimal compromise system design. The stability set of the first kind corresponding to the optimal system design has been defined and determined. The advantages of this approach are that it can be applied to any environment and enables the decision-maker to investigate real-world problems.

## Author Contributions

Hamiden Abd El-Wahed Khalifa conceptualized and designed the study, developed the theoretical framework, and led the analysis and interpretation of the data. Seyed Ahmad Edalatpanah contributed to the development of the mathematical models and provided insights into the fuzzy optimization techniques applied in the study. Sultan S. Alodhaibi assisted with the implementation of the numerical experiments, data collection, and the computational analysis. All authors contributed to the writing, reviewing, and final approval of the manuscript.

## Data Availability

The datasets generated and/or analyzed during the current study are available from the corresponding author on reasonable request.

## Conflicts of Interest

The authors declare no conflict of interest related to the publication of this paper.

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