Systemic Analytics

www.sa-journal.org

Syst. Anal. Vol. 3, No. 3 (2025) 222–231.

Paper Type: Original Article

Powerset-Theoretic Foundations for HyperAutomata and SuperHy-perAutomata

Takaaki Fujita

Independent Researcher, Shinjuku, Shinjuku-ku, Tokyo, Japan; fujita@math.hit-u.ac.jp.

Citation:

Received: 15 March 2025 Revised: 17 April 2025 Accepted: 20 May 2025	Fujita, T. (2025). Powerset-theoretic foundations for hyperautomata and superhyperautomata. *Systemic Analytics*, 3(3), 222-231

Abstract

Hyperstructure theory, grounded in the powerset construction, offers a rich framework for modeling relationships among elements in a set. Its extension, superhyperstructure, employs the n-th iterated powerset to capture multi-layered hierarchical interactions [1]. A Finite Automaton is a machine with a finite set of states, recognizing regular languages via transitions over input symbols (cf.[2, 3, 4]).
In this paper, we examine three computational models—classical finite automata, hyperautomata, and superhyperautomata—by providing concise mathematical definitions, illustrative examples, and key properties. Our presentation refines the powerset-based definitions originally given in [5], clarifying how hyperautomata generalize classical automata and how superhyperautomata further generalize hyperautomata. Through this unified treatment, we highlight the expressive hierarchies and closure properties that distinguish these models.

Keywords: Hyperstructure, Superhyperstructure, Powerset, HyperAutomata, SuperHyperAutomata.

1|Introduction

We begin by recalling the definitions of the well-established concept. Throughout this paper, all sets and structures under consideration are assumed to be finite. For further details on operations and related properties of each concept, the reader is referred to the relevant references as needed.

Corresponding Author: fujita@math.hit-u.ac.jp

https://doi.org/10.31181/sa202556

1.1|Hyperstructure and Superhyperstructure

A *Hyperstructure* is built upon the concept of a powerset, providing a framework to model relationships among elements within a set [6, 7, 8, 9]. Extending this idea, a *Superhyperstructure* leverages the n-th powerset to represent systems with multi-layered hierarchical relationships, enabling deeper abstractions and complexity [1, 10, 11, 12, 13]. Below, we formally define the n-th powerset as a foundation for these structures.

[Base Set] A *base set* S is the foundational set from which complex structures such as powersets and hyperstructures are derived. It is formally defined as:

$$S = \{x \mid x \text{ is an element within a specified domain}\}.$$

All elements in constructs like $\mathcal{P}(S)$ or $\mathcal{P}_n(S)$ originate from the elements of S.

[Powerset] The *powerset* of a set S, denoted $\mathcal{P}(S)$, is the collection of all possible subsets of S, including both the empty set and S itself. Formally, it is expressed as:

$$\mathcal{P}(S) = \{A \mid A \subseteq S\}.$$

[n-th Powerset] (cf.[14, 15, 16, 17])

The n-th powerset of a set H, denoted $P_n(H)$, is defined iteratively, starting with the standard powerset. The recursive construction is given by:

$$P_1(H) = P(H), \quad P_{n+1}(H) = P(P_n(H)), \quad \text{for } n \geq 1.$$

Similarly, the n-th non-empty powerset, denoted $P_n^*(H)$, is defined recursively as:

$$P_1^*(H) = P^*(H), \quad P_{n+1}^*(H) = P^*(P_n^*(H)).$$

Here, $P^*(H)$ represents the powerset of H with the empty set removed.

To establish a comprehensive framework for understanding Hyperstructures and Superhyperstructures, we present the following formal definitions and foundational concepts.

[Classical Structure] (cf.[14, 18]) A *Classical Structure* is a mathematical framework defined on a non-empty set H, characterized by one or more *Classical Operations* that adhere to specific *Classical Axioms*. Formally:

A *Classical Operation* is a function of the form:

$$\#_0 : H^m \to H,$$

where $m \geq 1$ denotes a positive integer, and H^m represents the m-fold Cartesian product of H. Examples include algebraic operations such as addition and multiplication in structures like groups, rings, and fields.

[Hyperstructure] (cf.[14, 18]) A *Hyperstructure* extends the concept of a Classical Structure by operating on the powerset of a base set. It is formally defined as:

$$\mathcal{H} = (\mathcal{P}(S), \circ),$$

where S is the base set, $\mathcal{P}(S)$ denotes its powerset, and $\circ$ is an operation defined for subsets within $\mathcal{P}(S)$.

[n-Superhyperstructure] (cf.[14, 18]) An *n-Superhyperstructure* generalizes the Hyperstructure by employing the n-th powerset of a base set. Formally, it is defined as:

$$\mathcal{SH}_n = (\mathcal{P}_n(S), \circ),$$

where S is the base set, $\mathcal{P}_n(S)$ represents the n-th powerset of S, and $\circ$ is an operation acting on elements of $\mathcal{P}_n(S)$.

1.2|Finite Automaton

A Finite Automaton is a machine with a finite set of states, recognizing regular languages via transitions over input symbols (cf.[2, 3, 4]). Let Σ be a finite input alphabet. Denote by Σ^* the set of all finite words over Σ, including the empty word ε.

[Nondeterministic Finite Automaton] (cf.[2, 3]) A nondeterministic finite automaton (NFA) is a quintuple

$$A = \big(Q,\, \Sigma,\, \delta,\, q_0,\, F\big),$$

where

- Q is a finite set of *states*,
- Σ is a finite *input alphabet*,
- $\delta : Q \times \Sigma \longrightarrow \mathcal{P}(Q)$ is the *transition function*,
- $q_0 \in Q$ is the *initial state*,
- $F \subseteq Q$ is the set of *accepting states*.

[Finite Automaton] Extended Transition Function: The transition function extends to $\delta^* : Q \times \Sigma^* \to \mathcal{P}(Q)$ by recursion:

$$\delta^*(q, \varepsilon) = \{q\}, \qquad \delta^*(q,\, wa) = \bigcup_{p \in \delta^*(q,w)} \delta(p, a),$$

for all $q \in Q$, $w \in \Sigma^*$, and $a \in \Sigma$.

Language Accepted: The *language* recognized by A is

$$L(A) \;=\; \{\, w \in \Sigma^* \mid \delta^*(q_0, w) \;\cap\; F \neq \varnothing \}.$$

A language $L \subseteq \Sigma^*$ is called *regular* if there exists an NFA A such that $L = L(A)$.

Deterministic Finite Automaton: A deterministic finite automaton (DFA) is a special case of an NFA in which the transition function $\delta : Q \times \Sigma \to Q$ is single-valued (equivalently, $\delta(q, a)$ is always a singleton). Every NFA admits an equivalent DFA recognizing the same language.

2|Review: HyperAutomaton

We introduce the concept of a *HyperAutomaton*, which is defined using the powerset construction. The formal definition is presented below (cf.[5]). Note that this definition has been refined based on the use of the powerset construction, improving upon the version originally given in [5].

[HyperAutomaton] Let V_0 be a finite set of *base states*. We define the *hyperstate space* as the powerset

$$\mathcal{P}(V_0) = \{\, X \mid X \subseteq V_0 \}.$$

[HyperAutomaton] (cf.[5]) A *HyperAutomaton* is a quintuple

$$\mathcal{H} = \big(Q,\, \Sigma,\, \delta,\, q_0,\, F\big),$$

where

- $Q \subseteq \mathcal{P}(V_0)$ is the finite set of *hyperstates*.
- Σ is a finite *input alphabet*.
- $\delta : Q \times \Sigma \longrightarrow \mathcal{P}(Q)$ is the *hypertransition function*, which assigns to each hyperstate and input symbol a set of successor hyperstates.
- $q_0 \in Q$ is the *initial hyperstate*.
- $F \subseteq Q$ is the set of *accepting hyperstates*.

[Finite-word hyperautomaton] The "finite-word hyperautomaton" (NFH) model presented in [19] (cf.[20, 21]) is *not* an instance of the HyperAutomaton defined above. In our HyperAutomaton

$$\mathcal{H} = (Q \subseteq \mathcal{P}(V_0),\, \Sigma,\, \delta,\, q_0,\, F),$$

states are subsets of a fixed base set V_0 and transitions map each hyperstate and input symbol to a set of successor hyperstates. By contrast, an NFH operates over *hyperwords* (sets of words) and uses

(1) a finite set of word-variables X,

(2) a quantifier prefix α over X, and

(3) an underlying NFA over $(\Sigma \cup \{\#\})^X$,

to accept or reject an entire set of words by universally or existentially quantifying over assignments. Its "hyper-ness" arises from quantification over word-sets rather than from powerset-valued transitions on a single base state set.

[Parity HyperAutomaton] Let the base state set be

$$V_0 = \{q_0,\, q_1\},$$

intended to represent "even" and "odd" parity, respectively. Define the hyperstate set

$$Q = \bigl\{\{q_0\},\, \{q_1\},\, \{q_0, q_1\}\bigr\} \;\subseteq\; \mathcal{P}(V_0),$$

and let the input alphabet be $\Sigma = \{1\}$. We introduce the hypertransition function $\delta : Q \times \Sigma \to \mathcal{P}(Q)$ by

$$\delta\bigl(\{q_0\}, 1\bigr) = \{\{q_1\}\},$$
$$\delta\bigl(\{q_1\}, 1\bigr) = \{\{q_0\}\},$$
$$\delta\bigl(\{q_0, q_1\}, 1\bigr) = \{\{q_0\},\, \{q_1\}\}.$$

Take the initial hyperstate to be

$$q_0' = \{q_0\},$$

and the accepting hyperstates

$$F = \{\{q_0\}\}.$$

Then the HyperAutomaton

$$\mathcal{H} = \bigl(Q,\ \Sigma,\ \delta,\ q_0',\ F\bigr)$$

accepts exactly those unary words 1^n for which n is even. Indeed, starting from $\{q_0\}$ each input symbol "1" toggles between $\{q_0\}$ and $\{q_1\}$, while from the combined state $\{q_0, q_1\}$ it can nondeterministically reach either parity.

[HyperAutomaton as Hyperstructure] Let V_0 be a finite base set and let

$$\mathcal{H} \;=\; (Q,\ \Sigma,\ \delta,\ q_0,\ F)$$

be a HyperAutomaton with $Q \subseteq \mathcal{P}(V_0)$. For each input symbol $a \in \Sigma$, define the *hyperoperation*

$$\circ_a : Q \;\longrightarrow\; \mathcal{P}(Q), \qquad X \;\mapsto\; X \circ_a \;=\; \delta(X, a).$$

Then the algebraic structure

$$(\, Q,\ \{\circ_a\}_{a \in \Sigma}\,)$$

is a Hyperstructure over the base set $Q = \mathcal{P}(V_0)$.

Proof: By definition of a Hyperstructure (cf.[14]), one requires:

(1) A nonempty set of "elements," here $Q \subseteq \mathcal{P}(V_0)$.

(2) A family of hyperoperations. We have exactly one hyperoperation $\circ_a$ for each $a \in \Sigma$.

(3) *Closure:* For any $X \in Q$ and $a \in \Sigma$, by the HyperAutomaton definition $\delta(X, a) \subseteq Q$, so

$$X \circ_a \;=\; \delta(X, a) \;\in\; \mathcal{P}(Q),$$

establishing closure under each $\circ_a$.

No further axioms are required for a general Hyperstructure. Hence $(Q, \{\circ_a\})$ satisfies the definition of a Hyperstructure. □

[HyperAutomaton Generalizes Finite Automaton] Let

$$A = (Q,\ \Sigma,\ \delta,\ q_0,\ F)$$

be a (possibly nondeterministic) finite automaton. Define

$$V_0 = Q, \quad Q' = \mathcal{P}(V_0), \quad q_0' = \{q_0\}, \quad F' = \left\{ X \subseteq Q : X \cap F \neq \varnothing \right\}.$$

For each $a \in \Sigma$, define

$$\delta' : Q' \times \Sigma \longrightarrow \mathcal{P}(Q'), \qquad \delta'(X, a) = \left\{ \delta(q, a) \mid q \in X \right\}.$$

Then

$$\mathcal{H} = \left(Q',\, \Sigma,\, \delta',\, q_0',\, F'\right)$$

is a HyperAutomaton which simulates A. In particular, for every word $w \in \Sigma^*$,

$$\delta^*(q_0, w) \cap F \neq \varnothing \iff \delta'^*(\{q_0\}, w) \cap F' \neq \varnothing.$$

Hence every finite automaton embeds as a HyperAutomaton.

Proof: Define the embedding $f : Q \to Q'$ by $f(q) = \{q\}$. Then:

- $f(q_0) = \{q_0\} = q_0'$.
- For each $q \in Q$ and $a \in \Sigma$, $f\big(\delta(q,a)\big) = \{\delta(q,a)\} \subseteq \delta'(\{q\}, a)$, so transitions agree under embedding.
- A state q is accepting ($q \in F$) iff $\{q\} \cap F' \neq \varnothing$.
- By induction on $|w|$, one checks $\delta'^*(\{q\}, w) = \left\{\delta^*(q, w)\right\}$.

Thus $\mathcal{H}$ recognizes exactly the same language as A, proving the generalization. □

3|Review: n-SuperHyperAutomaton

We introduce the concept of a *n-SuperHyperAutomaton*, which is defined using the powerset construction. The formal definition is presented below(cf.[5]). Note that this definition has been refined based on the use of the nth-powerset construction, improving upon the version originally given in [5].

[n-SuperHyperAutomaton] (cf.[5]) Let V_0 be a finite *base state set*. For each integer $k \geq 0$, define recursively

$$\mathcal{P}^0(V_0) = V_0, \qquad \mathcal{P}^{k+1}(V_0) = \mathcal{P}\big(\mathcal{P}^k(V_0)\big).$$

[n-SuperHyperAutomaton] (cf.[5]) An *n-SuperHyperAutomaton* is a quintuple

$$\mathcal{S}^{(n)} = \big(Q,\, \Sigma,\, \delta,\, q_0,\, F\big),$$

where

- $Q \subseteq \mathcal{P}^n(V_0)$ is the finite set of *n-superstates*.
- Σ is a finite *input alphabet*.
- $\delta : Q \times \Sigma \longrightarrow \mathcal{P}(Q)$ is the *superhypertransition function*, mapping each n-superstate and input symbol to a set of n-superstates.
- $q_0 \in Q$ is the *initial n-superstate*.
- $F \subseteq Q$ is the set of *accepting n-superstates*.

[Parity 2-SuperHyperAutomaton] Let the base state set be

$$V_0 = \{q_0,\, q_1\},$$

so that

$$\mathcal{P}(V_0) = \big\{\varnothing, \{q_0\}, \{q_1\}, \{q_0, q_1\}\big\}, \quad \mathcal{P}^2(V_0) = \mathcal{P}\big(\mathcal{P}(V_0)\big).$$

We choose the set of 2–superstates

$$Q = \left\{ \{\{q_0\}\},\ \{\{q_1\}\},\ \{\{q_0\}, \{q_1\}\} \right\} \subseteq \mathcal{P}^2(V_0),$$

and let the input alphabet be $\Sigma = \{a\}$. We define the superhypertransition function $\delta : Q \times \Sigma \to \mathcal{P}(Q)$ by

$$\delta\big(\{\{q_0\}\},\, a\big) = \{\{\{q_1\}\}\},$$
$$\delta\big(\{\{q_1\}\},\, a\big) = \{\{\{q_0\}\}\},$$
$$\delta\big(\{\{q_0\},\{q_1\}\},\, a\big) = \{\{\{q_0\}\},\, \{\{q_1\}\}\}.$$

We take as the initial 2–superstate

$$q_0' = \{\{q_0\}\},$$

and as the set of accepting superstates

$$F = \big\{\{\{q_0\}\}\big\}.$$

Then

$$\mathcal{S}^{(2)} = \big(Q,\ \Sigma,\ \delta,\ q_0',\ F\big)$$

accepts exactly those words a^n for which n is even. Intuitively, at the first level each symbol "a" toggles between the level-1 subsets $\{q_0\}$ and $\{q_1\}$, and at the second level the superstate records which subset was reached; the accepting superstate $\{\{q_0\}\}$ corresponds to even parity.

[Parity 3-SuperHyperAutomaton] Let the base state set be

$$V_0 = \{q_0,\, q_1\}.$$

Then

$$\mathcal{P}(V_0) = \{\varnothing, \{q_0\}, \{q_1\}, \{q_0, q_1\}\}, \quad \mathcal{P}^2(V_0) = \mathcal{P}\big(\mathcal{P}(V_0)\big), \quad \mathcal{P}^3(V_0) = \mathcal{P}\big(\mathcal{P}^2(V_0)\big).$$

We choose the set of 3–superstates

$$Q = \big\{\{\{\{q_0\}\}\},\ \{\{\{q_1\}\}\},\ \{\{\{q_0\}\},\{\{q_1\}\}\}\big\} \ \subseteq\ \mathcal{P}^3(V_0),$$

and let the input alphabet be $\Sigma = \{a\}$. Define the superhypertransition function $\delta : Q \times \Sigma \to \mathcal{P}(Q)$ by

$$\delta\big(\{\{\{q_0\}\}\},\, a\big) = \{\{\{\{q_1\}\}\}\},$$
$$\delta\big(\{\{\{q_1\}\}\},\, a\big) = \{\{\{\{q_0\}\}\}\},$$
$$\delta\big(\{\{\{q_0\}\},\{\{q_1\}\}\},\, a\big) = \{\{\{\{q_0\}\}\},\, \{\{\{q_1\}\}\}\}.$$

We take as the initial 3–superstate

$$q_0' = \{\{\{q_0\}\}\},$$

and as the accepting superstates

$$F = \big\{\{\{\{q_0\}\}\}\big\}.$$

Then

$$\mathcal{S}^{(3)} = \big(Q,\ \Sigma,\ \delta,\ q_0',\ F\big)$$

accepts exactly those words a^n for which n is even. At each level the symbol "a" toggles parity, and the nesting ensures that the accepting superstate $\{\{\{q_0\}\}\}$ corresponds to even occurrences of "a."

[n-SuperHyperAutomaton as n-Superhyperstructure] Let V_0 be a finite base set and fix an integer $n \geq 1$. Let

$$\mathcal{S}^{(n)} \ = \ \big(Q,\ \Sigma,\ \delta,\ q_0,\ F\big)$$

be an n-SuperHyperAutomaton with $Q \subseteq \mathcal{P}^n(V_0)$. For each $a \in \Sigma$, define

$$\circ_a : Q \ \longrightarrow\ \mathcal{P}(Q), \qquad X \ \mapsto\ X \circ_a \ =\ \delta(X, a).$$

Then

$$\big(Q,\ \{\circ_a\}_{a\in\Sigma}\big)$$

is an n-Superhyperstructure over the base set $\mathcal{P}^n(V_0)$.

Proof: Recall that an n-Superhyperstructure is defined on the n-th powerset $\mathcal{P}^n(V_0)$ with one or more hyperoperations (cf.[1]). We verify:

- *Underlying set:* Q is nonempty and $Q \subseteq \mathcal{P}^n(V_0)$ by assumption.
- *Family of hyperoperations:* We have one hyperoperation $\circ_a$ for each $a \in \Sigma$.

- *Closure:* For any $X \in Q$ and $a \in \Sigma$, the transition function gives
$$X \circ_a \;=\; \delta(X,a) \;\subseteq\; Q,$$
so each $\circ_a$ is closed on Q.

No additional axioms are necessary for the general notion of an n-Superhyperstructure. Therefore, $(Q, \{\circ_a\})$ is indeed an n-Superhyperstructure. □

[n-SuperHyperAutomaton Generalizes HyperAutomaton] Let
$$\mathcal{H} = (Q,\ \Sigma,\ \delta,\ q_0,\ F)$$
be a HyperAutomaton with $Q \subseteq \mathcal{P}(V_0)$. Fix an integer $n \geq 1$. Define
$$Q^{(n)} = \mathcal{P}^n(V_0),$$
$$q_0^{(n)} = \underbrace{\{\{\cdots\{}_{n-1 \text{ times}}\ q_0\ \underbrace{\}\cdots\}\}}_{n-1 \text{ times}},$$
$$F^{(n)} = \Big\{\ \underbrace{\{\{\cdots\{}_{n-1 \text{ times}}\ X\ \underbrace{\}\cdots\}\}}_{n-1 \text{ times}}\ \Big|\ X \in F\Big\}.$$
For each $a \in \Sigma$, set
$$\delta^{(n)} : Q^{(n)} \times \Sigma \to \mathcal{P}\big(Q^{(n)}\big), \quad \delta^{(n)}(Y,a) = \Bigg\{\ \underbrace{\{\{\cdots\{}_{n-1 \text{ times}}\ X'\ \underbrace{\}\cdots\}\}}_{n-1 \text{ times}}\ \Bigg|\ Y = \underbrace{\{\{\cdots\{}_{n-1 \text{ times}}\ X\ \underbrace{\}\cdots\}\}}_{n-1 \text{ times}}\Bigg\}.$$
Then
$$\mathcal{S}^{(n)} = \big(Q^{(n)},\, \Sigma,\, \delta^{(n)},\, q_0^{(n)},\, F^{(n)}\big)$$
is an n-SuperHyperAutomaton which simulates $\mathcal{H}$. Consequently, every HyperAutomaton embeds into an n-SuperHyperAutomaton.

Proof: Define the canonical nesting map $\varphi : Q \to Q^{(n)}$ by $\varphi(X) = \{\{\cdots\{X\}\cdots\}\}$ ($n-1$ nestings). Then:

(1) $\varphi(q_0) = q_0^{(n)}$ and $\varphi(F) = F^{(n)}$.

(2) For any $X \in Q$ and $a \in \Sigma$,
$$\delta^{(n)}\big(\varphi(X), a\big) = \{\varphi(X') \mid X' \in \delta(X,a)\} = \varphi\big(\delta(X,a)\big),$$
so transitions commute with φ.

(3) By induction on the length of $w \in \Sigma^*$, one shows $\delta^{(n)*}(\varphi(X), w) = \varphi\big(\delta^*(X,w)\big)$.

Thus $\mathcal{S}^{(n)}$ recognizes the same language as $\mathcal{H}$ under φ, establishing the embedding. □

[Extended Superhypertransition Function] Let $\mathcal{S}^{(n)} = (Q, \Sigma, \delta, q_0, F)$ be an n-SuperHyperAutomaton, and let $\delta^* : Q \times \Sigma^* \to \mathcal{P}(Q)$ be defined by
$$\delta^*(q,\varepsilon) = \{q\}, \qquad \delta^*(q,\, wa) = \bigcup_{r \in \delta^*(q,w)} \delta(r,a) \quad (w \in \Sigma^*,\, a \in \Sigma).$$
Then for all $q \in Q$ and all words $u, v \in \Sigma^*$ we have
$$\delta^*(q, uv) \;=\; \bigcup_{r \in \delta^*(q,u)} \delta^*(r,v).$$

Proof: We prove by induction on $|v|$.

Base case: If $v = \varepsilon$, then
$$\delta^*(q,\, u\varepsilon) = \delta^*(q,u) = \bigcup_{r \in \delta^*(q,u)} \{r\} = \bigcup_{r \in \delta^*(q,u)} \delta^*(r,\varepsilon).$$

Inductive step: Suppose the claim holds for all words of length $\le k$. Let $v = wa$ with $|w| = k$ and $a \in \Sigma$. Then

$$\delta^*(q,\, u\, wa) = \bigcup_{t \in \delta^*(q,\, u\, w)} \delta(t, a) \quad \text{(by definition of } \delta^*)$$

and by the inductive hypothesis,

$$\delta^*(q,\, u\, w) = \bigcup_{r \in \delta^*(q,u)} \delta^*(r, w).$$

Hence

$$\delta^*(q,\, u\, wa) = \bigcup_{r \in \delta^*(q,u)} \bigcup_{t \in \delta^*(r,w)} \delta(t, a) = \bigcup_{r \in \delta^*(q,u)} \delta^*(r,\, wa),$$

completing the induction. □

[Closure under Union] Let $\mathcal{S}_1^{(n)} = (Q_1, \Sigma, \delta_1, q_{01}, F_1)$ and $\mathcal{S}_2^{(n)} = (Q_2, \Sigma, \delta_2, q_{02}, F_2)$ be two n-SuperHyperAutomata with $Q_1 \cap Q_2 = \varnothing$. Then there exists an n-SuperHyperAutomaton $\mathcal{S}^{(n)}$ such that

$$L(\mathcal{S}^{(n)}) \;=\; L(\mathcal{S}_1^{(n)}) \;\cup\; L(\mathcal{S}_2^{(n)}).$$

Proof: Define

$$Q = Q_1 \cup Q_2 \cup \{q_0'\}, \quad q_0' \;\notin\; Q_1 \cup Q_2, \quad F = F_1 \cup F_2.$$

For each $a \in \Sigma$ define

$$\delta(q, a) = \begin{cases} \delta_1(q, a), & q \in Q_1, \\ \delta_2(q, a), & q \in Q_2, \\ \{q_{01},\, q_{02}\}, & q = q_0'. \end{cases}$$

Set $\mathcal{S}^{(n)} = (Q, \Sigma, \delta, q_0', F)$. Intuitively, from q_0' on the first symbol it nondeterministically "chooses" to simulate either $\mathcal{S}_1^{(n)}$ or $\mathcal{S}_2^{(n)}$. A word w is accepted by $\mathcal{S}^{(n)}$ iff it is accepted by at least one of the two original automata, proving closure under union. □

[Equivalence to Classical NFA] For every n-SuperHyperAutomaton $\mathcal{S}^{(n)} = (Q, \Sigma, \delta, q_0, F)$, there exists a (classical) nondeterministic finite automaton $\mathcal{A} = (Q, \Sigma, \delta', q_0, F)$ recognizing the same language, where

$$\delta'(q, a) = \delta(q, a) \quad \text{for all } q \in Q,\; a \in \Sigma.$$

Hence $\{\, L(\mathcal{S}^{(n)}) \mid \mathcal{S}^{(n)}$ is an n-SuperHyperAutomaton$\}$ is exactly the class of regular languages.

Proof: Simply observe that an n-SuperHyperAutomaton already uses a transition function $\delta : Q \times \Sigma \to \mathcal{P}(Q)$ just like an NFA. Define $\delta' = \delta$. Then by induction on word length one shows $\delta^*(q_0, w) \cap F \neq \varnothing$ in the n-SuperHyperAutomaton iff the NFA $\mathcal{A}$ accepts w. This establishes that every n-SuperHyperAutomaton recognizes a regular language, and conversely every NFA is trivially an n-SuperHyperAutomaton when viewed as having superstates of nesting level n. □

Author Contribution

The paper has been solely authored by the corresponding author at this stage.

Data Availability

This research is purely theoretical, involving no data collection or analysis. We encourage future researchers to pursue empirical investigations to further develop and validate the concepts introduced here.

Ethical Considerations

This work does not involve any experiments or studies involving human participants or animals, and therefore no ethical approvals were required.

Conflicts of Interest

The authors confirm that there are no conflicts of interest related to the research or its publication.

Research Integrity

The authors hereby confirm that, to the best of their knowledge, this manuscript is their original work, has not been published in any other journal, and is not currently under consideration for publication elsewhere at this stage.

Disclaimer (Note on Computational Tools)

No computer-assisted proof, symbolic computation, or automated theorem proving tools (e.g., Mathematica, SageMath, Coq, etc.) were used in the development or verification of the results presented in this paper. All proofs and derivations were carried out manually and analytically by the authors.

Disclaimer (Limitations and Claims)

The theoretical concepts presented in this paper have not yet been subject to practical implementation or empirical validation. Future researchers are invited to explore these ideas in applied or experimental settings. Although every effort has been made to ensure the accuracy of the content and the proper citation of sources, unintentional errors or omissions may persist. Readers should independently verify any referenced materials.

To the best of the authors' knowledge, all mathematical statements and proofs contained herein are correct and have been thoroughly vetted. Should you identify any potential errors or ambiguities, please feel free to contact the authors for clarification.

The results presented are valid only under the specific assumptions and conditions detailed in the manuscript. Extending these findings to broader mathematical structures may require additional research. The opinions and conclusions expressed in this work are those of the authors alone and do not necessarily reflect the official positions of their affiliated institutions.

References

[1] Florentin Smarandache. Superhyperstructure & neutrosophic superhyperstructure, 2024. Accessed: 2024-12-01.

[2] Dominique Perrin. Finite automata. In *Formal Models and Semantics*, pages 1–57. Elsevier, 1990.

[3] BF Melnikov. The complete finite automaton. *International Journal of Open Information Technologies*, 5(10):9–17, 2017.

[4] Michael O Rabin and Dana Scott. Finite automata and their decision problems. *IBM journal of research and development*, 3(2):114–125, 1959.

[5] Takaaki Fujita and Florentin Smarandache. Uncertain automata and uncertain graph grammar. *Neutrosophic Sets and Systems*, 74:128–191, 2024.

[6] Piergiulio Corsini and Violeta Leoreanu. *Applications of hyperstructure theory*, volume 5. Springer Science & Business Media, 2013.

[7] Bijan Davvaz and Thomas Vougiouklis. *Walk Through Weak Hyperstructures, A: Hv-structures*. World Scientific, 2018.

[8] Michal Novák, Štepán Křehlík, and Kyriakos Ovaliadis. Elements of hyperstructure theory in uwsn design and data aggregation. *Symmetry*, 11(6):734, 2019.

[9] Adel Al-Odhari. Neutrosophic powerset and neutrosophic hyperstructure of neutrosophic set of three types. 2025.

[10] Florentin Smarandache. *The SuperHyperFunction and the Neutrosophic SuperHyperFunction (revisited again)*, volume 3. Infinite Study, 2022.

[11] Takaaki Fujita. Toward a unified framework for knot theory, hyperknot theory, and superhyperknot theory via superhyperstructures. *Neutrosophic Knowledge*, 6:55–71, 2025.

[12] Ajoy Kanti Das, Rajat Das, Suman Das, Bijoy Krishna Debnath, Carlos Granados, Bimal Shil, and Rakhal Das. A comprehensive study of neutrosophic superhyper bci-semigroups and their algebraic significance. *Transactions on Fuzzy Sets and Systems*, 8(2):80, 2025.

[13] Adel Al-Odhari. A brief comparative study on hyperstructure, super hyperstructure, and n-super superhyperstructure. *Neutrosophic Knowledge*, 6:38–49, 2025.

[14] Florentin Smarandache. Foundation of superhyperstructure & neutrosophic superhyperstructure. *Neutrosophic Sets and Systems*, 63(1):21, 2024.

[15] Florentin Smarandache. *SuperHyperFunction, SuperHyperStructure, Neutrosophic SuperHyperFunction and Neutrosophic SuperHyperStructure: Current understanding and future directions*. Infinite Study, 2023.

[16] Takaaki Fujita. Hyperalgorithms & superhyperalgorithms: A unified framework for higher-order computation. *Prospects for Applied Mathematics and Data Analysis*, 4(1):36–49, 2024.

[17] Takaaki Fujita. Antihypergeometry, neutrohypergeometry, and superhypergeometry. *Advancing Uncertain Combinatorics through Graphization, Hyperization, and Uncertainization: Fuzzy, Neutrosophic, Soft, Rough, and Beyond*, page 568, 2025.

[18] F. Smarandache. Introduction to superhyperalgebra and neutrosophic superhyperalgebra. *Journal of Algebraic Hyperstructures and Logical Algebras*, 2022.

[19] Borzoo Bonakdarpour and Sarai Sheinvald. Finite-word hyperlanguages. *Information and Computation*, 295:104944, 2023.

[20] Borzoo Bonakdarpour and Sarai Sheinvald. Automata for hyperlanguages. *arXiv preprint arXiv:2002.09877*, 2020.

[21] Hadar Frenkel and Sarai Sheinvald. Realizable and context-free hyperlanguages. *arXiv preprint arXiv:2209.10306*, 2022.