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Information Geometric Analysis of the Dynamics of Transient M/M/∞ Queue Manifold

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Abstract

From a differential geometric perspective, Information Geometry (IG) aims to characterise the structure of statistical geodesic models. The research done for this paper offers a novel method for modelling the IG of a queuing system. From the perspective of IG, the manifold of the temporary M/M/∞ queue is described in this context. The Fisher Information Matrix (FIM) as well as the Inverse of FIM, (IFIM) of transient M/M/∞ Queue Manifold (QM) are devised. In addition to that, new results that uncovered the significant impact of stability of M/M/∞ QM on the existence of IFIM are obtained. Moreover, the Geodesic Equations (GEs) of motion of the coordinates of the underlying transient M/M/∞ are obtained. Also, it is revealed that stable M/M/∞ QM is developable (i.e., has a zero Gaussian curvature) and has a non-zero Ricci Curvature Tensor (RCT). The Information Matrix Exponential (IME) is devised. Also, it is shown that stability of the devised IME enforces the instability of M/M/∞ QM. More interestingly, novel relativistic info-geometric queueing theoretic links are revealed. A summary combined with future research work are given.

Keywords: Transient M/M/ ∞ queueing system, Information geometry, Statistical manifold, Queue manifold, Geodesic equations of motion, Ricci curvature, Einstein tensor, Stress energy tensor, Riemannian metric, Fisher information matrix, Inverse fisher information matrix, Threshold theorem.

1 | Introduction

Numerous study areas, including statistical inference, stochastic control, and neural networks, have extensively used Information Geometry (IG) [1]. In other words, the goal of IG is to use statistics to apply the methods of Differential Geometry (DG).

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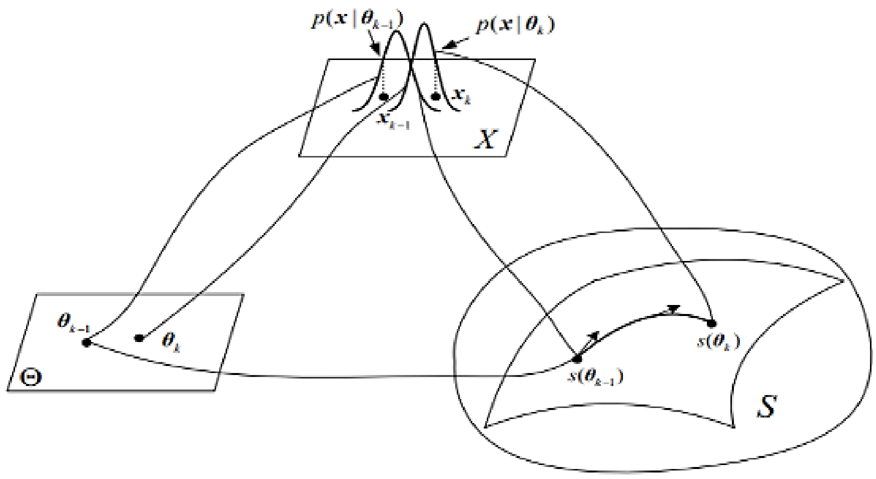

Fig. 1. The parameter inference $\widehat{\theta}$ for an infinite-dimensional manifold [2].

A manifold [2] is a topological finite dimensional Cartesian space, $\mathbb{R}^n$, where one has an infinite-dimensional manifold. In *Fig. 1*, the parameter inference $\hat{\theta}$ of a model from data can be interpreted as a decision-making problem [3].In [1], [4], the exponential distribution families were investigated with many variations.

In this research, the geometric structure of the stable M/G/1 Queue Manifold (QM) is studied. The IG of a stable M/D/1 queue has only been the subject of one research study [3], in which a geometric structure was introduced on the set of M/D/1 queues by utilising the features of queue length routes. This perspective drove the original research path in this study, which links information matrix theories and IG to provide a fresh understanding of the transient M/M/∞ queue.

In this work, scalar curvature [3] quantifies the deviation for a geodesic ball's volume from that of a Euclidean ball with the same radius, while Ricci Curvature (RC) [3] assesses the deviation of the Riemannian Metric (RM) from the standard Euclidean Metric (EM).

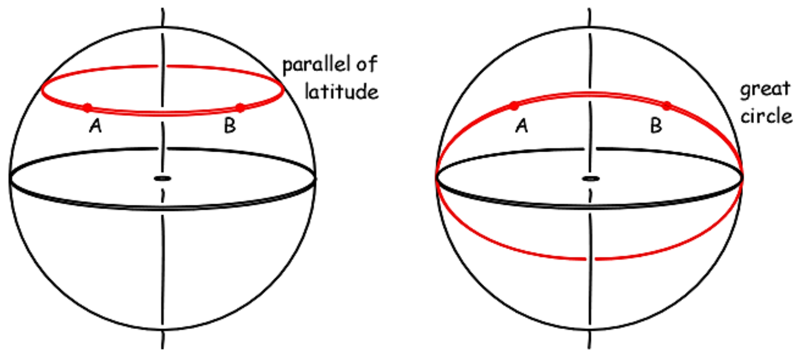

Fig. 2. Geometric representation of geodesics on curved surfaces [3].

2 | Main Definitions

2.1 | Main Definition on IG

Definition 1 (Statistical Manifold (SM)). We call $M = \{p(x;\theta) \mid \theta = (\theta^1, \theta^2, \ldots, \theta^n) \epsilon \mathbb{R}^n\}$ a manifold [1] of n-dimensional distribution with coordinate system $(\theta^1, \theta^2, \ldots, \theta^n)$.

Definition 2 (potential function). The potential function, $\Psi(\theta)$ [1] is the distinguished negative function of the coordinates alone of $(\mathcal{L}(x;\theta) = \ln(p(x;\theta)))$. Fundamentally, $\Psi(\theta)$ is the part of $(-\mathcal{L}(x;\theta))$ which only contains $(\theta^1, \theta^2, \ldots, \theta^n)$.

Definition 3. Fisher Information Matrix (FIM), or $[g_{ij}]$) [1] reads:

$$[g_{ij}] = \left[\frac{\partial^2}{\partial \theta^i \partial \theta^j}(\Psi(\theta))\right], i, j \in [1,2,3, \dots, n\}. \quad (1)$$

Definition 4. Inverse of FIM, (IFIM), or $[g^{ij}]$) reads as [5]

$$[g^{ij}] = ([g_{ij}])^{-1} = \frac{\text{adj}[g_{ij}]}{\Delta}, \Delta = \det[g_{ij}]. \quad (2)$$

The corresponding arc length function is

$$(ds)^2 = \sum_{i,j=1}^{n} g_{ij} (d\theta^i)(d\theta^j). \quad (3)$$

Definition 5 (α-connection). For each $\alpha \epsilon \mathbb{R}$, the α (or $\nabla^{(\alpha)}$)-connection [5] reads:

$$\Gamma_{ij,k}^{(\alpha)} = (\tfrac{1-\alpha}{2})(\partial_i \partial_j \partial_k(\Psi(\theta))). \quad (4)$$

$\Psi(\theta)$ (Definition), $\partial_i = \frac{\partial}{\partial \theta_i}$.

Definition 6.

I. The Geodesic Equations (GEs) of manifold M with coordinate system $\theta = (\theta_1, \theta_2, .., \theta_n)$ are defined by Mageed [1].

$$\frac{d^2\theta^k}{dt^2} + \Gamma_{ij}^{k(0)} \left(\frac{d\theta^i}{dt}\right)\left(\frac{d\theta^j}{dt}\right) = 0, i, j = 1,2, \dots, n, \Gamma_{ij}^{k(\alpha)} = \Gamma_{ij,s}^{(\alpha)} g^{sk}. \quad (5)$$

II. The GEs [1] that characterize the curves that minimize the length/energy between two arbitrary points on a smooth manifold M.

III. The total energy [6] of a path $\theta = \theta(t)$, between $t = a$ and $t = b$, can be defined in terms of a Lagrangian function $L = L\left(\theta, \frac{d\theta}{dt}\right)$, as follows:

$$E(\theta) = \int_{t_1}^{t_2} L(\theta(t), \theta^{\cdot}(t))\, dt. \quad (6)$$

The path $\theta = \theta(t)$ that minimizes the total energy $E = E(\theta)$ necessarily satisfies the Euler–Lagrange equations. Here these take the form of Lagrange's equations of motion.

$$\frac{d}{dt}\left(\frac{\partial^2 L}{\partial(\frac{d\theta_j}{dt})}\right) - \frac{\partial}{\partial \theta_j} = 0, \quad (7)$$

for each $j = 1,2, \dots, n$. In the following, we use g^{ij} (the inverse FIM) to denote symmetric positive matrix g_{ij} (FIM) (where $i, j = 1,2, .., n$) so that:

$$\sum_{k=1}^{n} g^{ik} g_{kj} = \delta_{ij} = f(x) = \begin{cases} 1, & i = j, \\ 0, & \text{otherwise}. \end{cases} \quad (8)$$

Lemma 1 (GEs). Lagrange's equations [6] of motion for the Lagrangian,

$$L = L\left(\theta, \frac{d\theta}{dt}\right), \quad (9)$$

are given in local coordinates by the system of ordinary differential equations.

$$\frac{d^2\theta_i}{dt^2} + \sum_{j,k=1}^{n} \Gamma_{j,k}^{i} \left(\frac{d\theta_j}{dt}\right)\left(\frac{d\theta_k}{dt}\right) = 0, \quad (10)$$

where the quantities $\Gamma^{i}_{j,k}$ are known as the Christoffel symbols and for $i, j, k = 1,2, .., n$.

By the above definition, it is clear that the GEs are interpreted physically as the Information Geometric Equations of Motion (IGEMs), or the Relativitstic Equations of Motion (REMs), or the Riemannian equations of motion. At this stage, it is important to note that this report is the first time ever which sets ground breaking discovery of the IG analysis of transient queueing systems in comparison to that of non-time dependent queueing systems, namely IG analysis of stable queueues [2], [7].

Definition 7.

I. Under the θ coordinate system, the $\alpha -$ curvature RiemannianTensors, $R^{(\alpha)}_{ijkl}$ [1] are defined by

$$R^{(\alpha)}_{ijkl} = \left[\left(\partial_j\Gamma^{s(\alpha)}_{ik} - \partial_i\Gamma^{s(\alpha)}_{jk}\right)g_{sl} + \left(\Gamma^{(\alpha)}_{j\beta,l}\Gamma^{\beta(\alpha)}_{ik} - \Gamma^{(\alpha)}_{i\beta,l}\Gamma^{\beta(\alpha)}_{jk}\right)\right], i, j, k, l, s, \beta = 1,2,3, \ldots, n, \quad (11)$$

where $\Gamma^{k(\alpha)}_{ij} = \Gamma^{(\alpha)}_{ij,s}g^{sk}$.

II. The $\alpha -$ Ricci curvatures (Ricci Tensors) $R^{(\alpha)}_{ik}$ are determined by [1]

$$R^{(\alpha)}_{ik} = R^{(\alpha)}_{ijkl}g^{jl}. \quad (12)$$

III. The $\alpha -$ sectional curvatures $K^{(\alpha)}_{ijij}$ are defined by [1]

$$K^{(\alpha)}_{ijij} = \frac{R^{(\alpha)}_{ijij}}{(g_{ii})(g_{jj}) - (g_{ij})^2}, i, j = 1,2, \ldots, n. \quad (13)$$

Specifically, if $n = 2$, the $\alpha -$ sectional curvature $K^{(\alpha)}_{1212} = K^{(\alpha)}$ is called $\alpha -$ Gaussian curvature and is given by [1]

$$K^{(\alpha)} = \frac{R^{(\alpha)}_{1212}}{\det(g_{ij})}. \quad (14)$$

IV. The Riemannian Tensor [1] is simply contracted to create the Ricci Tensor [8].

V. The degree to which a geodesic ball's volume on the surface varies from a geodesic ball's volume in Euclidean space is known as the Ricci Curvature Tensor (RCT) [9] of an oriented Riemannian Manifold M. The evolution of volumes under the geodesic flow is contracted by the RCT [10]. The Bonnet-Myers theorem [10] states that when RC is positive, the Riemannian manifold has a smaller diameter and is more positively curved than a sphere.

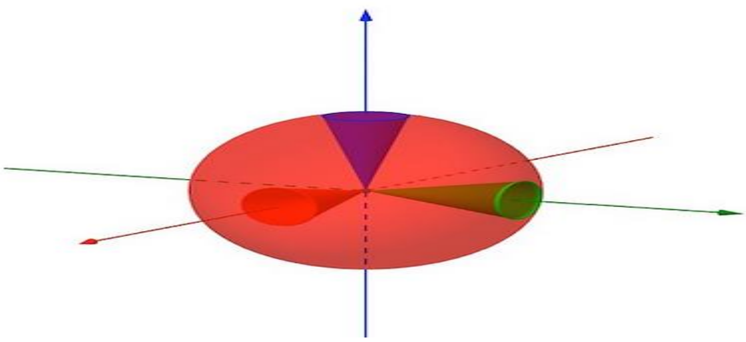

Fig. 3. Ricci Curvature Tensor visualization [11].

Definition 8. A certain category of ruled surfaces called developable surfaces can be mapped onto a plane surface without causing any deformation to the curves; each curve drawn from such a surface onto the flat plane retains its original shape [12].

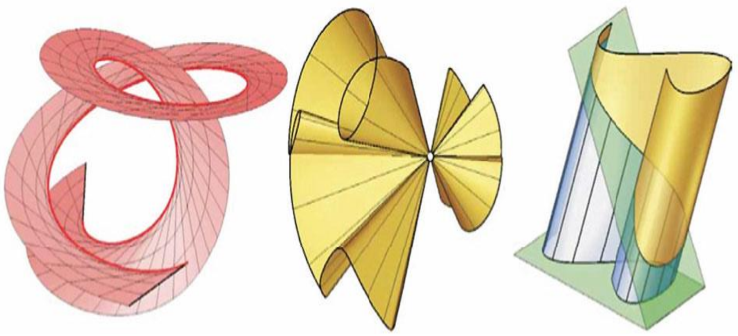

Fig. 4. Developable surfaces [12].

Definition 9 ([13]). A function is well-defined if it gives the same result when the representation of the input is changed without changing the value of the input.

Definition 10 ([14]).

I. Function f is said to be one-to-one, or injective, if and only if $f(x) = f(y)$ implies $x = y$ for all x, y in the domain of f. A function is said to be an injection if it is one-to-one. Alternative: a function is one-to-one if and only if $f(x) \neq f(y)$, whenever $x \neq y$. This is the contrapositive of the definition.

II. A function f from A to B is called onto, or surjective, if and only if for every $b \in B$ there is an element $a \in A$ such that $f(a) = b$. Alternative: all co-domain elements are covered.

III. A function f is called a bijection if it is both one-to-one (injection) and onto (surjection).

Theorem 1. [15].

Let f be a function that is defined and differentiable on an open interval (c,d).

$$\text{If } f'(x) > 0 \text{ for all } x \in (c, d)\text{, then f is increasing on } (c, d). \quad \textbf{(15)}$$

$$\text{If } f'(x) < 0 \text{ for all } x \in (c, d)\text{, then f is decreasing on } (c, d). \quad \textbf{(16)}$$

Taylor expansions [16] are widely used to approximate functions by expansions. We have for all x around zero,

$$\ln(1 + x) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n}. \quad \textbf{(17)}$$

$$\frac{1}{1 + x} = \sum_{n=0}^{\infty} (-1)^n x^n. \quad \textbf{(18)}$$

2.4 | Important Inequalities [17]

I. Chebyshev inequality.

$$\frac{x}{1 + x} \leq \ln(1 + x) \leq x \text{ for } x \geq -1. \quad \textbf{(19)}$$

Eq. (19) can be rewritten as

$$1 - \frac{1}{x} \leq \ln(x) \leq x - 1 \text{ for } x \geq -1. \quad \textbf{(20)}$$

II. Lehmer inequality

$$\frac{x^n}{n!} < \frac{x^n}{n!} + 1 \leq e^x \text{ for all } x, n > 0. \tag{21}$$

3 | The Fim and Its Inverse for the Transient M/M/∞ QM

In queueing theory, a discipline within the mathematical theory of probability, the M/M/∞ queue [18] is a multi-server queueing model where every arrival experience immediate service and does not wait.

The transient probability of the M/M/∞ queueing system with Poissson arrival rate ξ [19] and exponentially distributed service time with mean $\frac{1}{\xi}$ and arrival rate λ is given by

$$p_n(t) = \frac{[\frac{\lambda}{\xi}(1-e^{-\xi t})]^n}{n!} \exp\{-\frac{\lambda}{\xi}(1-e^{-\xi t})\}, \qquad n = 0,1,2,\dots. \tag{22}$$

It has been pointed out by Conolly and Langaris [20] that one of the best-known Bessel function forms for the time-dependent state probabilities in M/M/1/∞ queueing system is given by

$$Q_n(t) = \frac{e^{-(\lambda+\mu)(t)}}{\mu t} (\frac{\lambda}{\mu})^n \sum_{m=n+1}^{\infty} m\beta^{-m} I_m(\omega t). \tag{23}$$

Theorem 2. For the transient formalism of M/M/∞ queueing system, we have

$[g_{ij}]$ (*Eq. (1)*) reads as

$$[g_{ij}] = \begin{bmatrix} 0 & a & b \\ a & d_1 & g \\ l & h & r \end{bmatrix}, \tag{24}$$

where

$$a = \frac{1}{\xi^2}\left([1+t\xi]e^{-t\xi} - 1\right). \tag{25}$$

$$b = \frac{1}{\xi^2}\left(-\dot{\xi} + \xi e^{-t\xi}[t\dot{\xi} + \xi + \dot{\xi}]\right). \tag{26}$$

$$d_1 = \frac{\lambda}{\xi^3}\left(2 - [1+t\xi][t\xi+2]e^{-t\xi}\right). \tag{27}$$

$$g = \frac{1}{\xi^3}\left([\dot{\lambda}\xi - 2\lambda\dot{\xi}] + e^{-t\xi}[\dot{\lambda}\xi - 2\lambda t\xi\dot{\xi} + \lambda t\xi^2 - 2\lambda\xi - \lambda\xi^2 t^2\dot{\xi} - \lambda t\xi^3]\right). \tag{28}$$

$$l = \frac{1}{\xi^2}\left(-\dot{\xi} + \xi e^{-t\xi}[t\dot{\xi} + \xi]\right). \tag{29}$$

$$h = \frac{(\dot{\lambda}\xi - 2\lambda\dot{\xi})}{\xi^3}\left[(1+\xi t)e^{-t\xi} - 1\right]. \tag{30}$$

$$\begin{aligned} r = \frac{1}{\xi^3}\big((\ddot{\lambda}\xi^2 - \lambda\xi\ddot{\xi} - 2\dot{\lambda}\xi\dot{\xi} + 2\lambda\dot{\xi}^2) \\ + e^{-t\xi}[\xi(\lambda t\xi\dot{\xi} + \lambda t\dot{\xi}^2 + \lambda t\xi\ddot{\xi} + 3\lambda\xi t\dot{\xi} + \dot{\lambda}\xi^2 + \lambda\ddot{\xi} - \ddot{\lambda}\xi) - (\xi[t\dot{\xi} + \xi] \\ - 2\dot{\xi})(\lambda\xi t\dot{\xi} + \lambda\xi^2 + \lambda\dot{\xi} - \dot{\lambda}\xi)]]\big). \end{aligned} \tag{31}$$

Provided that, refers to the temporal derivative $\frac{d}{dt}$.

IFIM reads as

$$[g^{ij}] = \frac{\operatorname{adj}[g_{ij}]}{\Delta} = \begin{bmatrix} A & B & L \\ D_1 & E & F \\ G & H & I \end{bmatrix}, \Delta = \det([g_{ij}]) = \left(-a(lg - ar) + b(ah - ld_1)\right), \tag{32}$$

where

$$A = \frac{(rd_1 - gh)}{\Delta}. \quad (33)$$

$$B = \frac{(hb - ar)}{\Delta}. \quad (34)$$

$$L = \frac{(ag - bd_1)}{\Delta}. \quad (35)$$

$$D_1 = \frac{(lg - ar)}{\Delta}. \quad (36)$$

$$E = \frac{(-lb)}{\Delta}. \quad (37)$$

$$F = \frac{(ab)}{\Delta}. \quad (38)$$

$$G = \frac{(ah - ld_1)}{\Delta}. \quad (39)$$

$$H = \frac{(la)}{\Delta}. \quad (40)$$

$$I = \frac{(-a^2)}{\Delta}. \quad (41)$$

Proof: following *Eq. (22)*, we have

$$\mathcal{L}(x;\theta) = \ln\left(p_n(x;\theta)\right) = \ln\left(\frac{\left[\frac{\lambda}{\xi}\left(1-e^{-\xi t}\right)\right]^n}{n!}\exp\left\{-\frac{\lambda}{\xi}\left(1-e^{-\xi t}\right\}\right) = \ln\left(\frac{\left[\frac{\lambda}{\xi}\left(1-e^{-\xi t}\right)\right]^n}{n!}\right) - \left\{\frac{\lambda}{\xi}\left(1-e^{-\xi t}\right\}, \quad (42)$$

$$\theta = (\theta_1, \theta_2, \theta_3) = (\lambda, \xi, t).$$

Hence, we have

$$\Psi(\theta) = \frac{\lambda}{\xi}\left(1 - e^{-\xi t}\right). \quad (43)$$

Thus,

$$\partial_1 = \frac{\partial \Psi}{\partial \lambda} = \frac{1}{\xi}\left(1 - e^{-\xi t}\right), \partial_2 = \frac{\partial \Psi}{\partial \xi} = \frac{\lambda}{\xi^2}\left((1+\xi t)e^{-\xi t} - 1\right), \partial_3 = \frac{\partial \Psi}{\partial t} = \frac{1}{\xi^2}\left((\lambda\cdot\xi - \lambda\xi^{\cdot}) + (\lambda\xi\xi^{\cdot}t + \lambda\xi^2 - \lambda\cdot\xi + \lambda\xi^{\cdot})e^{-\xi t}\right). \quad (44)$$

$$\partial_1 \partial_1 = 0. \quad (45)$$

$$\partial_1 \partial_2 = \partial_2 \partial_1 = \frac{1}{\xi^2}\left([1 + t\xi]e^{-t\xi} - 1\right). \quad (46)$$

$$\partial_2 \partial_2 = \frac{\lambda}{\xi^3}\left(2 - [1 + t\xi][t\xi + 2]e^{-t\xi}\right). \quad (47)$$

$$\partial_2 \partial_3 = \frac{1}{\xi^3}\left([2\lambda\xi^{\cdot} - \lambda\cdot\xi] + e^{-t\xi}[\lambda\cdot t\xi^2 - 2\lambda t\xi\xi^{\cdot} + \lambda t\xi^2 - \lambda\xi - \lambda\xi^2 t^2\xi^{\cdot} - \lambda t\xi^3 + 2\lambda\cdot\xi - 2\lambda\xi^{\cdot}]\right). \quad (48)$$

$$\partial_3 \partial_2 = \frac{(\lambda\cdot\xi - 2\lambda\xi^{\cdot})}{\xi^3}\left[(1 + \xi t)e^{-t\xi} - 1\right] \neq \partial_2 \partial_3. \quad (49)$$

$$\partial_1 \partial_3 = \frac{1}{\xi^2}\left(-\xi^{\cdot} + \xi e^{-t\xi}[t\xi^{\cdot} + \xi + \xi^{\cdot}]\right). \quad (50)$$

$$\partial_3 \partial_1 = \frac{1}{\xi^2}\left(-\xi^{\cdot} + \xi e^{-t\xi}[t\xi^{\cdot} + \xi]\right) \neq \partial_1 \partial_3. \quad (51)$$

$$\partial_3 \partial_3 = \quad (52)$$

$\frac{1}{\xi^3}((\ddot{\lambda}\xi^2 - \lambda\xi\ddot{\xi} - 2\dot{\lambda}\xi\dot{\xi} + 2\lambda\dot{\xi}^2) + e^{-t\xi}[\xi(\lambda t\xi\dot{\xi} + \lambda t\dot{\xi}^2 + \lambda t\xi\ddot{\xi} + 3\lambda\xi t\dot{\xi} + \dot{\lambda}\xi^2 + \lambda\ddot{\xi} - \ddot{\lambda}\xi) - (\xi[t\dot{\xi} + \xi] - 2\dot{\xi})(\lambda\xi t\dot{\xi} + \lambda\xi^2 + \lambda\dot{\xi} - \dot{\lambda}\xi)]]).$

Therefore, the FIM, is obtained (*Eq. (24)*).

We write IFIM as

$$[g^{ij}] = [g_{ij}]^{-1} = \frac{\operatorname{adj}[g_{ij}]}{\Delta} = \frac{\text{Transpose}(\text{Cov}[g_{ij}])}{\Delta} = \frac{1}{\Delta}\text{transpose}\left(\text{Cov}\begin{bmatrix} 0 & a & b \\ a & d_1 & g \\ l & h & r \end{bmatrix}\right). \tag{53}$$

$$\left(\text{Cov}\begin{bmatrix} 0 & a & b \\ a & d_1 & g \\ l & h & r \end{bmatrix}\right) = \begin{bmatrix} (rd_1 - gh) & (lg - ar) & (ah - ld_1) \\ (hb - ar) & (-lb) & (la) \\ (ag - bd_1) & (ab) & (-a^2) \end{bmatrix}. \tag{54}$$

Thus,

$$[g^{ij}] = \frac{1}{\Delta}\text{transpose}\left(\text{Cov}\begin{bmatrix} (rd_1 - gh) & (lg - ar) & (ah - ld_1) \\ (hb - ar) & (-lb) & (la) \\ (ag - bd_1) & (ab) & (-a^2) \end{bmatrix}\right) = \begin{bmatrix} A & B & L \\ D_1 & E & F \\ G & H & I \end{bmatrix}. \text{ (Eqs. (33)-(39))}$$

Notably, the symmetry requirement should hold for FIM. Meanwhile, it is observed that FIM (*Eq. (24)* of *Theorem 2*) is not symmetric, since $\partial_2\,\partial_3 \neq \partial_3\,\partial_2$ and $\partial_1\,\partial_3 \neq \partial_3\,\partial_1$. This raises many open problems on such novel phenomenon and shows the significant impact of time as a coordinate in the potential function (*Eq. (44)*). it is inevitable to investigate under what conditions will FIM be symmetric. The following theorem answers this question.

Theorem 3. For the transient formalism of M/M/∞ queueing system, as time approaches infinity, FIM (*Eq. (24)* of *Theorem 2*) is symmetric. The converse statement is not always true.

Proof: it is sufficient to prove that if $t \to \infty$,

$$\partial_2\,\partial_3 = \partial_3\,\partial_2 \text{ and } \partial_1\,\partial_3 = \partial_3\,\partial_1. \tag{55}$$

Clearly if t→∞, it holds that

$$\lim_{t\to\infty}\partial_2\,\partial_3 = \lim_{t\to\infty}\partial_3\,\partial_2 = \frac{(\dot{\lambda}\xi - 2\lambda\dot{\xi})}{\xi^3}\lim_{t\to\infty}\partial_1\,\partial_3 = \lim_{t\to\infty}\partial_3\,\partial_1 = -\frac{\dot{\xi}}{\xi^2}.$$

Consequently, the corresponding FIM given by

$$[g_{ij}]_{t\to\infty} = \begin{bmatrix} 0 & -\frac{1}{\xi^2} & -\frac{\dot{\xi}}{\xi^2} \\ -\frac{1}{\xi^2} & \frac{\lambda}{\xi^3} & \frac{1}{\xi^3}(2\lambda\dot{\xi} - \dot{\lambda}\xi) \\ -\frac{\dot{\xi}}{\xi^2} & \frac{1}{\xi^3}(2\lambda\dot{\xi} - \dot{\lambda}\xi) & \frac{1}{\xi^3}((\ddot{\lambda}\xi^2 - \lambda\xi\ddot{\xi} - 2\dot{\lambda}\xi\dot{\xi} + 2\lambda\dot{\xi}^2) \end{bmatrix}, \tag{56}$$

which is symmetric.

To investigate the necessity requirement: let $\partial_2\,\partial_3 = \partial_3\,\partial_2$ and $\partial_1\,\partial_3 = \partial_3\,\partial_1$. This generates the following two sets of equations.

$$\frac{1}{\xi^3}\left([2\lambda\dot{\xi} - \dot{\lambda}\xi] + e^{-t\xi}[\dot{\lambda}t\xi^2 - 2\lambda t\xi\dot{\xi} + \lambda t\xi^2 - \lambda\xi - \lambda\xi^2 t^2\dot{\xi} - \lambda t\xi^3 + 2\dot{\lambda}\xi - 2\lambda\dot{\xi}]\right)$$
$$= \frac{(\dot{\lambda}\xi - 2\lambda\dot{\xi})}{\xi^3}[(1 + \xi t)e^{-t\xi} - 1], \tag{57}$$

and

$$\frac{1}{\xi^2}\left(-\xi^{\cdot}+\xi e^{-t\xi}[t\xi^{\cdot}+\xi+\xi^{\cdot}]\right) = \frac{1}{\xi^2}\left(-\xi^{\cdot}+\xi e^{-t\xi}[t\xi^{\cdot}+\xi]\right), \quad (58)$$

Eq. (58) reduces to:

$$\frac{e^{-t\xi}}{\xi^3}[\lambda^{\cdot}t\xi^2-2\lambda t\xi\xi^{\cdot}+\lambda t\xi^2-\lambda\xi-\lambda\xi^2t^2\xi^{\cdot}-\lambda t\xi^3+2\lambda^{\cdot}\xi-2\lambda\xi^{\cdot}] = \frac{(\lambda^{\cdot}\xi-2\lambda\xi^{\cdot})}{\xi^3}\left[(1+\xi t)e^{-t\xi}\right]. \quad (59)$$

This implies:

$$\frac{e^{-t\xi}}{\xi^3}=0, \quad (60)$$

or

$$[\lambda^{\cdot}t\xi^2-2\lambda t\xi\xi^{\cdot}+\lambda t\xi^2-\lambda\xi-\lambda\xi^2t^2\xi^{\cdot}-\lambda t\xi^3+2\lambda^{\cdot}\xi-2\lambda\xi^{\cdot}] = (\lambda^{\cdot}\xi-2\lambda\xi^{\cdot}))(1+\xi t). \quad (61)$$

Similarly, *Eq. (59)* reduces to *Eq. (61)* and

$$\xi[t\xi^{\cdot}+\xi+\xi^{\cdot}] = \xi[t\xi^{\cdot}+\xi]. \quad (62)$$

Clearly, it follows from *Eq. (61)*

$$\text{Either } t\to\infty \text{ or } \xi\to\infty. \quad (63)$$

Eq. (62) generates the following differential equation.

$$[\lambda^{\cdot}t\xi^2-2\lambda t\xi\xi^{\cdot}+\lambda t\xi^2-\lambda\xi-\lambda\xi^2t^2\xi^{\cdot}-\lambda t\xi^3+2\lambda^{\cdot}\xi-2\lambda\xi^{\cdot}] - (\lambda^{\cdot}\xi-2\lambda\xi^{\cdot}))(1+\xi t)=0. \quad (64)$$

It could be verified that $\lambda=0$ and $\xi=0$ are two solutions of *Eq. (65)*.

The reader can check that *Eq. (63)* reduces to

$$\xi[t\xi^{\cdot}+\xi+\xi^{\cdot}] = \xi[t\xi^{\cdot}+\xi], or \quad \xi\xi^{\cdot}=0. \quad (65)$$

It could be verified that $\xi=\text{constant}$ is the closed form solution of *Eq. (66)*. Proof completed.

Theorem 3. For the transient formalism of M/M/∞ queueing system, as time approaches infinity, $[g^{ij}]_{t\to\infty}$ (*Eq. (33)* of *Theorem 2*) does not exist if and only if one of following requirements is satisfied:

$$\xi\to\infty, \quad (66)$$

or whenever λ,ξ are solutions of the following differential equation.

$$[\lambda\xi\xi^{\cdot\cdot}+2\lambda\xi^{\cdot 2}-\lambda^{\cdot\cdot}\xi^2]=0. \quad (67)$$

Proof: following Matrix Algebra, $[g^{ij}]_{t\to\infty}$ does not exist if and only if $\Delta[g_{ij}]_{t\to\infty}=0$, $[g_{ij}]_{t\to\infty}$(*Eq. (57)*). Therefore, $[g^{ij}]_{t\to\infty}$ will never exist if and only if

$$\frac{1}{\xi^2}\left[\frac{1}{\xi^3}\left((\lambda^{\cdot\cdot}\xi^2-\lambda\xi\xi^{\cdot\cdot}-2\lambda^{\cdot}\xi\xi^{\cdot}+2\lambda\xi^{\cdot 2})\left(-\frac{1}{\xi^2}\right)+\frac{1}{\xi^3}(2\lambda\xi^{\cdot}-\lambda^{\cdot}\xi)\left(\frac{\xi^{\cdot}}{\xi^2}\right)\right]-\frac{\xi^{\cdot}}{\xi^2}\left[\frac{\lambda}{\xi^3}\left(\frac{\xi^{\cdot}}{\xi^2}\right)-\frac{1}{\xi^3}(2\lambda\xi^{\cdot}-\lambda^{\cdot}\xi)\left(\frac{1}{\xi^2}\right)\right]=0,$$

equivalently

$$\frac{1}{\xi^7}[\lambda\xi\xi^{\cdot\cdot}+2\lambda\xi^{\cdot 2}-\lambda^{\cdot\cdot}\xi^2]=0. \quad (68)$$

Hence, $\Delta[g_{ij}]_{t\to\infty}=0$ if and only if

$$\frac{1}{\xi^7} = 0, \tag{69}$$

or

$$[\lambda\xi\ddot{\xi} + 2\lambda\dot{\xi}^2 - \ddot{\lambda}\xi^2] = 0. \quad \textit{(Eq. (68))}$$

Clearly, *Eq. (70)* implies $\xi \to \infty$ (*Eq. (67)*).

It could be verified that $\lambda = 0$ and $\xi = 0$ are two solutions of *Eq. (68)*. Moreover $\xi = 1$ in *Eq. (68)* implies $\ddot{\lambda} = 0$, which has the closed form solution, $\lambda = c_1 + c_2 t$ a family of families of temporal straight lines. As time reaches infinity, the arrival rate in this case becomes infinite. This completes the proof.

In the following section, the components of α (or $\nabla^{(\alpha)}$)- connection are obtained and it will be shown that the algebraic structure of this connection is no longer symmetric as in the non-time dependent case. This unexpected phenomenon is justified by strong impact of the time coordinate relating to the time-dependency. Also, it is shown that for sufficiently large temporal values (i.e., $t \to \infty$), the generated special case of α (or $\nabla^{(\alpha)}$)- connection is symmetric. All these genuine contributions are new to the knowledge to both IG and queueing theorists. These calculated expressions are needed to obtain the corresponding GEs of the parametric coordinates of M/M/∞ QM.

4 | The α (or $\nabla^{(\alpha)}$)-Connection of the Transient M/M/∞ QM

4.1 | The Obtained $\Gamma_{ij,k}^{(\alpha)}$ Expressions (Definition 5) of the Transient M/M/∞ QM

By *Definition 5*, the reader can check that

$$\Gamma_{11,1}^{(\alpha)} = 0 = \Gamma_{11,2}^{(\alpha)} = \Gamma_{12,1}^{(\alpha)} = \Gamma_{21,1}^{(\alpha)} = \Gamma_{31,1}^{(\alpha)}. \tag{70}$$

$$\Gamma_{11,3}^{(\alpha)} = 0. \tag{71}$$

$$\Gamma_{13,1}^{(\alpha)} = 0 = \Gamma_{11,3}^{(\alpha)}. \tag{72}$$

$$\Gamma_{13,2}^{(\alpha)} = \frac{(1-\alpha)\dot{\xi}}{\xi^3}\left[1 - (1 + \xi t)e^{-t\xi}\right]. \tag{73}$$

$$\Gamma_{12,3}^{(\alpha)} = \frac{(1-\alpha)}{2\xi^3}\left(\dot{\xi} + e^{-t\xi}[t\xi^2 - 2t\xi\dot{\xi} - \xi - \xi^2 t^2 \dot{\xi} - t\xi^3 - 2\dot{\xi}]\right) \neq \Gamma_{13,2}^{(\alpha)}. \tag{74}$$

$$\Gamma_{12,2}^{(\alpha)} = \frac{(1-\alpha)}{2\xi^3}\left(2 - [1 + t\xi][t\xi + 2]e^{-t\xi}\right). \tag{75}$$

$$\Gamma_{13,3}^{(\alpha)} = \frac{(1-\alpha)}{2\xi^3}((2\dot{\xi}^2 - \xi\ddot{\xi}) + e^{-t\xi}[\xi(t\xi\dot{\xi} + t\dot{\xi}^2 + t\xi\ddot{\xi} + 3\xi t\dot{\xi} + \ddot{\xi}) - (\xi[t\dot{\xi} + \xi] - 2\dot{\xi})(\xi t\dot{\xi} + \xi^2 + \dot{\xi})]]). \tag{76}$$

$$\Gamma_{21,2}^{(\alpha)} = \Gamma_{22,1}^{(\alpha)} = \frac{(1-\alpha)}{2\xi^3}(2 - e^{-t\xi}\left(t^2\xi^2 + 2(1 + t\xi)\right). \tag{77}$$

Engaging the same approach, the remaining $\Gamma_{ij,k}^{(\alpha)}$ expressions can be obtained.

5 | The α (or $\nabla^{(\alpha)}$)-Connection of the Transient M/M/∞ QM When Time Is Infinite

5.1 | Geometry of M/M/∞ QM as $t \to \infty$

Now, we are in a situation to reveal the significant temporal impact on the overall IG analysis of the underlying M/M/∞ QM. This would be clear as time approaches infinity and in this phase; the FIM will be symmetric which not the case was before as time was not infinite. This also reflects upon the paths of motion of the corresponding GEs of motion for each coordinate of the underlying M/M/∞ QM.

Setting $t \to \infty$ in *Eqs. (25)-(31)*, we have

$$a = -\frac{1}{\xi^2}, \tag{78}$$

$$b = -\frac{\xi^{\cdot}}{\xi^2}, \tag{79}$$

$$d_1 = \frac{2\lambda}{\xi^3}. \tag{80}$$

$$g = \frac{[\lambda^{\cdot}\xi - 2\lambda\xi^{\cdot}]}{\xi^3}, \tag{81}$$

$$l = -\frac{\xi^{\cdot}}{\xi^2}, \tag{82}$$

$$h = \frac{(2\lambda\xi^{\cdot} - \lambda^{\cdot}\xi)}{\xi^3}, \tag{83}$$

$$r = \frac{(\lambda^{\cdot\cdot}\xi^2 - \lambda\xi\xi^{\cdot\cdot} - 2\lambda^{\cdot}\xi\xi^{\cdot} + 2\lambda\xi^{\cdot 2})}{\xi^3}, \tag{84}$$

with

$$A = \frac{(rd_1 - gh)}{\Delta}, \tag{85}$$

$$B = \frac{(hb - ar)}{\Delta}, \tag{86}$$

$$L = \frac{(ag - bd_1)}{\Delta}, \tag{87}$$

$$D_1 = \frac{(lg - ar)}{\Delta}, \tag{88}$$

$$E = \frac{(-lb)}{\Delta}, \tag{89}$$

$$F = \frac{(ab)}{\Delta}, \tag{90}$$

$$G = \frac{(ah - ld_1)}{\Delta}, \tag{91}$$

$$H = \frac{(la)}{\Delta}, \tag{92}$$

$$I = \frac{(-a^2)}{\Delta}, \Delta = \det([g_{ij}]) = (-a(lg - ar) + b(ah - ld_1)). \tag{93}$$

5.2 | The $\Gamma_{ij}^{k(\alpha)}$ Expressions of the Arrival Rate Coordinate, λ of the Transient M/M/∞ QM Corresponding to Infinite Temporal Values

$$\Gamma_{11}^{1(\alpha)} = 0. \tag{94}$$

$$\Gamma_{12}^{1(\alpha)} = \frac{(1-\alpha)}{2\xi^3}(2D_1 + \xi^{\cdot}G). \tag{95}$$

$$\Gamma_{21}^{1(\alpha)} = \frac{(1-\alpha)}{2\xi^3}(2D_1 + 2G\xi^{\cdot}). \tag{96}$$

$$\Gamma_{13}^{1(\alpha)} = \frac{(1-\alpha)}{2\xi^3}(2D_1 + (2\xi^{\cdot 2} - \xi\xi^{\cdot\cdot})G). \tag{97}$$

Hence,

$$\Gamma_{13}^{1(0)} = \frac{1}{2\xi^3}(2D_1 + (2\dot{\xi}^2 - \xi\ddot{\xi})G). \quad (98)$$

$$\Gamma_{31}^{1(\alpha)} = \frac{(1-\alpha)}{2\xi^3}(2\xi\dot{} D_1 + (2\dot{\xi}^2 - \xi\ddot{\xi})G). \quad (99)$$

$$\Gamma_{22}^{1(\alpha)} = \frac{(1-\alpha)}{2\xi^4}((2A\xi - 6\lambda D_1 + (2\dot{\lambda}\xi - 6\lambda\dot{\xi})G). \quad (100)$$

$$\Gamma_{23}^{1(\alpha)} = \frac{(1-\alpha)}{2\xi^4}(2A\dot{\xi} + (2\dot{\lambda}\xi - 6\lambda\dot{\xi})D_1 + [(\xi(2\xi\ddot{\lambda} - \lambda\ddot{\xi} - 2\dot{\lambda}\dot{\xi}) - 3(\ddot{\lambda}\xi^2 - \lambda\xi\ddot{\xi} - 2\dot{\lambda}\xi\dot{\xi} + 2\lambda\dot{\xi}^2)]G). \quad (101)$$

$$\Gamma_{32}^{1(\alpha)} = \frac{(1-\alpha)}{2\xi^4}(2A\xi\dot{\xi} - 6\dot{\xi}D_1 + (\xi[2\lambda\ddot{\xi} + \dot{\lambda}\dot{\xi} - \ddot{\lambda}\xi] - 3\dot{\xi}[2\lambda\dot{\xi} - \dot{\lambda}\xi])G). \quad (102)$$

$$\Gamma_{33}^{1(\alpha)} = \frac{(1-\alpha)}{2\xi^4}\left(\xi A(2\dot{\xi}^2 - \xi\ddot{\xi}) + \left((-\ddot{\lambda}\xi + 2\lambda\ddot{\xi} + \dot{\lambda}\dot{\xi})\xi + 3(\dot{\lambda}\xi - 2\lambda\dot{\xi})\dot{\xi}\right)D_1 + \left(\xi(\dddot{\lambda}\xi^2 - \lambda\xi\dddot{\xi} - 3\dot{\lambda}\xi\ddot{\xi} + 3\lambda\dot{\xi}\ddot{\xi}) - 3\dot{\xi}(\ddot{\lambda}\xi^2 - \lambda\xi\ddot{\xi} - 2\dot{\lambda}\xi\dot{\xi} + 2\lambda\dot{\xi}^2)\right)G\right). \quad (103)$$

Similarly, the remaining components at infinite temporal can be obtained.

6|The IMEs of the Coordinates the Transient M/M/∞ QM When Time Is Infinite

6.1|The IMEs of the Arrival Rate Coordinate, λ of the Transient M/M/∞ QM

The Information Matrix Exponentials (IMEs) (*Eq. (7)*) corresponding to the arrival rate coordinate, λ of the transient M/M/∞ QM are

$$\frac{d^2\theta^1}{dt^2} + \Gamma_{ij}^{1(0)}\left(\frac{d\theta^i}{dt}\right)\left(\frac{d\theta^j}{dt}\right) = 0, i, j = 1,2,3.$$

Now, we are in a situation of trying to find the path of motion of family of families of IMEs corresponding to the arrival rate coordinate, λ.

$$[\frac{d^2\lambda}{dt^2} + (\frac{1}{2\xi^3}(2D_1 + \dot{\xi}G) + \frac{1}{2\xi^3}(2D_1 + 2G\dot{\xi}))(\frac{d\lambda}{dt})(\frac{d\xi}{dt}) + [\frac{1}{2\xi^3}(2D_1 + (2\dot{\xi}^2 - \xi\ddot{\xi})G))\frac{1}{2\xi^3}(2D_1 + (2\dot{\xi}^2 - \xi\ddot{\xi})G))](\frac{d\lambda}{dt})(\frac{dt}{dt}) + \frac{1}{2\xi^4}[((2A\xi - 6\lambda D_1 + (2\dot{\lambda}\xi - 6\lambda\dot{\xi})G)](\frac{d\xi}{dt})^2 + [\frac{1}{2\xi^4}(2A\dot{\xi} + (2\dot{\lambda}\xi - 6\lambda\dot{\xi})D_1 + [(\xi(2\xi\ddot{\lambda} - \lambda\ddot{\xi} - 2\dot{\lambda}\dot{\xi}) - 3(\ddot{\lambda}\xi^2 - \lambda\xi\ddot{\xi} - 2\dot{\lambda}\xi\dot{\xi} + 2\lambda\dot{\xi}^2)]G) + \frac{1}{2\xi^4}(2A\xi\dot{\xi} - 6\dot{\xi}D_1 + (\xi[2\lambda\ddot{\xi} + \dot{\lambda}\dot{\xi} - \ddot{\lambda}\xi] - 3\dot{\xi}[2\lambda\dot{\xi} - \dot{\lambda}\xi])G)](\frac{d\xi}{dt})(\frac{dt}{dt}) + (\frac{1}{2\xi^4}(\xi A(2\dot{\xi}^2 - \xi\ddot{\xi}) + \left((-\ddot{\lambda}\xi + 2\lambda\ddot{\xi} + \dot{\lambda}\dot{\xi})\xi + 3(\dot{\lambda}\xi - 2\lambda\dot{\xi})\dot{\xi}\right)D_1 + \left(\xi(\dddot{\lambda}\xi^2 - \lambda\xi\dddot{\xi} - 3\dot{\lambda}\xi\ddot{\xi} + 3\lambda\dot{\xi}\ddot{\xi}) - 3\dot{\xi}(\ddot{\lambda}\xi^2 - \lambda\xi\ddot{\xi} - 2\dot{\lambda}\xi\dot{\xi} + 2\lambda\dot{\xi}^2)\right)G)\left(\frac{dt}{dt}\right)^2] = 0. \quad (104)$$

Without any loss of generality, let $\xi = 1$, so *Eq. (107)* reduces to

$$[\dddot{\lambda}G + (1 - D_1)\ddot{\lambda} + 2\dot{\lambda}D_1] = 0. \quad (105)$$

At $\xi = 1$, *Eqs. (79)-(86)* reduce to

$$a = -1. \quad (106)$$

$$b = 0. \quad (107)$$

$$d_1 = 2\lambda. \quad (108)$$

$$g = \dot{\lambda}. \quad (109)$$

$$l = 0. \quad (110)$$

$$h = -\dot{\lambda}. \quad (111)$$

$$r = \ddot{\lambda}. \tag{112}$$

This implies that *Eq. (91)* and *Eq. (94)* reduce to

$$D_1 = \frac{(lg - ar)}{\Delta} = \frac{\ddot{\lambda}}{\ddot{\lambda}} = 1. \tag{113}$$

$$G = \frac{(ah - ld_1)}{\Delta} = \frac{\dot{\lambda}}{\ddot{\lambda}}. \tag{114}$$

$$\Delta = \ddot{\lambda}. \tag{115}$$

Consequently,

$$\dddot{\lambda}\left(\frac{\dot{\lambda}}{\ddot{\lambda}}\right) + 2\dot{\lambda} = 0 = \dot{\lambda}(\dddot{\lambda} + 2\,\ddot{\lambda}) = 0. \tag{116}$$

Eq. (119) implies $\dot{\lambda} = 0$, with a closed form solution $\lambda = \text{constant}$. This explains that $\lambda = \text{constant}$ characterize the path of motion of the arrival rate coordinate of M/M/∞ QM. By *Eq. (119)*, $(\dddot{\lambda} + 2\,\ddot{\lambda}) = 0$. Let $\lambda = \kappa e^{\Upsilon t}$, $\Upsilon^3 + 2\Upsilon^2 = 0 = \Upsilon^2(2 + \Upsilon) = 0$ implying $\Upsilon_{1,2,3} = 0,0,-2$. Therefore, we have the closed form solutions represented by the family of families,

$$\lambda = \kappa_1 + \kappa_2 t + \kappa_2 e^{-2t}. \tag{117}$$

This supports a strong evidence that the corresponding paths of motion of the arrival rate for the Poisson arrival rate $\xi = 1$ are devised by family of families $\lambda = \text{constant}$. This explains that $\lambda = \text{constant}$ or $\lambda = \kappa_1 + \kappa_2 t + \kappa_2 e^{-2t}$.

6.2 | The IMEs of the Poisson Arrival Ratecoordinate, ξ of the Transient M/M/∞ QM

The IMEs (*Eq. (7)*) corresponding to the Poisson arrival ratecoordinate, ξ of the transient M/M/∞ QM are

$$\frac{d^2\theta^2}{dt^2} + \Gamma_{ij}^{1(0)}\left(\frac{d\theta^i}{dt}\right)\left(\frac{d\theta^j}{dt}\right) = 0, i, j = 1,2,3.$$

Now, we are in a situation of trying to find the path of motion of family of families of IMEs corresponding to the Poisson arrival rate coordinate, ξ. As $t \to \infty$, we have

$$\Gamma_{11}^{2(\alpha)} = 0. \tag{118}$$

$$\Gamma_{12}^{2(\alpha)} = \frac{(1-\alpha)}{2\xi^3}(2E + \dot{\xi}H). \tag{119}$$

$$\Gamma_{21}^{2(\alpha)} = \frac{(1-\alpha)}{2\xi^3}(2E + 2H\dot{\xi}). \tag{120}$$

$$\Gamma_{13}^{2(\alpha)} = \frac{(1-\alpha)}{2\xi^3}(2E + (2\dot{\xi}^2 - \xi\ddot{\xi})H). \tag{121}$$

$$\Gamma_{31}^{2(\alpha)} = \frac{(1-\alpha)}{2\xi^3}(2\dot{\xi}E + (2\dot{\xi}^2 - \xi\ddot{\xi})H). \tag{122}$$

$$\Gamma_{22}^{2(\alpha)} = \frac{(1-\alpha)}{2\xi^4}((2B\xi - 6\lambda E + (2\dot{\lambda}\xi - 6\lambda\dot{\xi})H). \tag{123}$$

$$\Gamma_{23}^{2(\alpha)} = \frac{(1-\alpha)}{2\xi^4}(2B\dot{\xi} + (2\dot{\lambda}\xi - 6\lambda\dot{\xi})E + [(\xi(2\xi\ddot{\lambda} - \lambda\ddot{\xi} - 2\dot{\lambda}\dot{\xi}) - 3(\ddot{\lambda}\xi^2 - \lambda\xi\ddot{\xi} - 2\dot{\lambda}\xi\dot{\xi} + 2\lambda\dot{\xi}^2)]H). \tag{124}$$

$$\Gamma_{32}^{2(\alpha)} = \frac{(1-\alpha)}{2\xi^4}(2B\xi\dot{\xi} - 6\dot{\xi}E + (\xi[2\lambda\ddot{\xi} + \dot{\lambda}\dot{\xi} - \ddot{\lambda}\xi] - 3\dot{\xi}[2\lambda\dot{\xi} - \dot{\lambda}\xi])H). \tag{125}$$

$$\Gamma_{33}^{2(\alpha)} = \frac{(1-\alpha)}{2\xi^4}\Big(\xi B(2\dot{\xi}^2 - \xi\ddot{\xi}) + \big((-\ddot{\lambda}\xi + 2\lambda\ddot{\xi} + \dot{\lambda}\dot{\xi})\xi + 3(\dot{\lambda}\xi - 2\lambda\dot{\xi})\dot{\xi}\big)E + \big(\xi(\dddot{\lambda}\xi^2 - \lambda\xi\dddot{\xi} - 3\dot{\lambda}\xi\ddot{\xi} + 3\lambda\dot{\xi}\ddot{\xi}) - 3\dot{\xi}(\ddot{\lambda}\xi^2 - \lambda\xi\ddot{\xi} - 2\dot{\lambda}\xi\dot{\xi} + 2\lambda\dot{\xi}^2)\big)H\Big). \quad (126)$$

Setting $\lambda = 1$, we have

$$a = \frac{-1}{\xi^2}. \quad (127)$$

$$b = -\frac{\dot{\xi}}{\xi^2}. \quad (128)$$

$$d_1 = \frac{2}{\xi^3}. \quad (129)$$

$$g = \frac{-2\dot{\xi}}{\xi^3}. \quad (130)$$

$$l = -\frac{\dot{\xi}}{\xi^2}. \quad (131)$$

$$h = \frac{2\dot{\xi}}{\xi^3}. \quad (132)$$

$$r = \frac{(-\xi\ddot{\xi} + 2\dot{\xi}^2)}{\xi^3}, \Delta = \det([g_{ij}] = \frac{-\ddot{\xi}}{\xi^6}. \quad (133)$$

$$\Gamma_{11}^{2(0)} = 0. \quad (134)$$

$$\Gamma_{12}^{2(0)} = \frac{1}{2\xi^3}(2E + \dot{\xi}H). \quad (135)$$

$$\Gamma_{21}^{2(0)} = \frac{1}{2\xi^3}(2E + 2H\dot{\xi}). \quad (136)$$

$$\Gamma_{13}^{2(0)} = \frac{1}{2\xi^3}(2E + (2\dot{\xi}^2 - \xi\ddot{\xi})H). \quad (137)$$

$$\Gamma_{31}^{2(0)} = \frac{1}{2\xi^3}(2\xi\dot{E} + (2\dot{\xi}^2 - \xi\ddot{\xi})H). \quad (138)$$

$$\Gamma_{22}^{2(0)} = \frac{1}{\xi^4}((B\xi - 3E + (-3\dot{\xi})H). \quad (139)$$

$$\Gamma_{23}^{2(0)} = \frac{1}{2\xi^4}(2B\dot{\xi} + (-6\dot{\xi})E + [(\xi(-\ddot{\xi}) - 3(\xi^2 - \xi\ddot{\xi} + 2\dot{\xi}^2)]H). \quad (140)$$

$$\Gamma_{32}^{2(0)} = \frac{1}{2\xi^4}(2B\xi\dot{\xi} - 6\dot{\xi}E + (\xi[2\ddot{\xi}] - 3\dot{\xi}[2\dot{\xi}])H). \quad (141)$$

$$\Gamma_{33}^{2(0)} = \frac{1}{2\xi^4}\Big(\xi B(2\dot{\xi}^2 - \xi\ddot{\xi}) + \big((2\ddot{\xi})\xi + 3(-2\dot{\xi})\dot{\xi}\big)E + \big(\xi(-\xi\dddot{\xi}) - 3\dot{\xi}(-\xi\ddot{\xi} + 2\dot{\xi}^2)\big)H\Big). \quad (142)$$

$$B = \frac{(hb-ar)}{\Delta} = \frac{((\frac{2\dot{\xi}}{\xi^3})(-\frac{\dot{\xi}}{\xi^2}) + (\frac{(-\xi\ddot{\xi}+2\dot{\xi}^2)}{\xi^5}))}{\frac{-\ddot{\xi}}{\xi^6}} = \xi^2. \quad (143)$$

$$E = \frac{(-lb)}{\Delta} = -\frac{\frac{\dot{\xi}^2}{\xi^4}}{\frac{-\ddot{\xi}}{\xi^6}} = \frac{\xi^2\dot{\xi}^2}{\ddot{\xi}}. \quad (144)$$

$$H = \frac{(la)}{\Delta} = \frac{\frac{\dot{\xi}}{\xi^4}}{\frac{-\ddot{\xi}}{\xi^6}} = -\frac{\xi^2\dot{\xi}}{\ddot{\xi}}. \quad (145)$$

The resulting IMEs corresponding to $\lambda = 1$ is represented by family of families of paths of motions given by

$$[\frac{d^2\xi}{dt^2} + (\frac{1}{\xi^4}((B\xi - 3E + (-3\dot{\xi})H))\left(\frac{d\xi}{dt}\right)^2 + [\frac{1}{2\xi^4}(2B\dot{\xi} + (-6\dot{\xi})E + [(\xi(-\ddot{\xi}) - 3(\xi^2 - \xi\ddot{\xi} + 2\dot{\xi}^2)]H) + \frac{1}{2\xi^4}(2B\xi\dot{\xi} - 6\dot{\xi}E + (\xi[2\ddot{\xi}] - 3\dot{\xi}[2\dot{\xi}])H)](\frac{d\xi}{dt}) + \left[\frac{1}{2\xi^4}\left(\xi B(2\dot{\xi}^2 - \xi\ddot{\xi}) + \left((2\ddot{\xi})\xi + 3(-2\dot{\xi})\dot{\xi}\right)E + \left(\xi(-\xi\dddot{\xi}) - 3\dot{\xi}(-\xi\ddot{\xi} + 2\dot{\xi}^2)\right)H\right)\right] = 0. \quad (146)$$

By *Eqs. (107)-(148)*, we have *Eq. (149)* in the form:

$$[\frac{d^2\xi}{dt^2} + (\frac{1}{\xi^4}(\left(\xi^3 - 3\left(\frac{\xi^2\dot{\xi}^2}{\ddot{\xi}}\right) + (-3\dot{\xi})\left(-\frac{\xi^2\dot{\xi}}{\ddot{\xi}}\right)\right))\left(\frac{d\xi}{dt}\right)^2 + [\frac{1}{2\xi^4}(2\xi^2\dot{\xi} + (-6\dot{\xi})(\frac{\xi^2\dot{\xi}^2}{\ddot{\xi}}) + [(\xi(-\ddot{\xi}) - 3(\xi^2 - \xi\ddot{\xi} + 2\dot{\xi}^2)](-\frac{\xi^2\dot{\xi}}{\ddot{\xi}})) + \frac{1}{2\xi^4}(2\xi^3\dot{\xi} - 6\dot{\xi}(\frac{\xi^2\dot{\xi}^2}{\ddot{\xi}}) + (\xi[2\ddot{\xi}] - 3\dot{\xi}[2\dot{\xi}])(-\frac{\xi^2\dot{\xi}}{\ddot{\xi}}))](\frac{d\xi}{dt}) + \left[\frac{1}{2\xi^4}\left(\xi^3(2\dot{\xi}^2 - \xi\ddot{\xi}) + \left((2\ddot{\xi})\xi + 3(-2\dot{\xi})\dot{\xi}\right)(\frac{\xi^2\dot{\xi}^2}{\ddot{\xi}}) + \left(\xi(-\xi\dddot{\xi}) - 3\dot{\xi}(-\xi\ddot{\xi} + 2\dot{\xi}^2)\right)(-\frac{\xi^2\dot{\xi}}{\ddot{\xi}})\right)\right] = 0. \quad (147)$$

The complicated *Eq. (150)* can be rewritten in the form:

$$[\frac{d^2\xi}{dt^2} + \frac{1}{\xi}\left(\frac{d\xi}{dt}\right)^2 + \frac{1}{2\xi^2}[(2\dot{\xi} + (-6\dot{\xi})(\frac{\dot{\xi}^2}{\ddot{\xi}}) + [2\xi\ddot{\xi} - \xi^2 - 6\dot{\xi}^2](-\frac{\dot{\xi}}{\ddot{\xi}})) + (2\xi\dot{\xi} - 6(\frac{\dot{\xi}^3}{\ddot{\xi}}) + (2\xi\ddot{\xi} - 6\dot{\xi}^2)(-\frac{\dot{\xi}}{\ddot{\xi}}))](\frac{d\xi}{dt}) + \frac{1}{2\xi^2}\left[\left(\xi(2\dot{\xi}^2 - \xi\ddot{\xi}) + (2\xi\ddot{\xi} - 6\dot{\xi}^2)(\frac{\dot{\xi}^2}{\ddot{\xi}}) + (3\xi\dot{\xi}\ddot{\xi} - \xi^2\dddot{\xi} - 6\dot{\xi}^3)(-\frac{\dot{\xi}}{\ddot{\xi}})\right)\right] = 0. \quad (148)$$

We can put *Eq. (151)* into the more compact form:

$$[2\xi^2\ddot{\xi}^2 + 2\xi\ddot{\xi}\dot{\xi}^2 + [(2\dot{\xi}\ddot{\xi} + (-6\dot{\xi})(\dot{\xi}^2) + [2\xi\ddot{\xi} - \xi^2 - 6\dot{\xi}^2](-\dot{\xi})) + (2\xi\ddot{\xi}\dot{\xi} - 6(\dot{\xi}^3) + (2\xi\ddot{\xi} - 6\dot{\xi}^2)(-\xi))](\dot{\xi}) + \left[\left(\xi\ddot{\xi}(2\dot{\xi}^2 - \xi\ddot{\xi}) + (2\xi\ddot{\xi} - 6\dot{\xi}^2)(\dot{\xi}^2) + (3\xi\dot{\xi}\ddot{\xi} - \xi^2\dddot{\xi} - 6\dot{\xi}^3)(-\dot{\xi})\right)\right] = 0. \quad (149)$$

The family of families of constant paths of motion of the Poisson arrival rate given b y, $\xi = \text{constant}$ provide a closed form solution of *Eq. (151)* η ς.

6.3 | The IMEs of the Temporal Coordinate, t of the Transient M/M/∞ QM

The IMEs (*Eq. (7)*) corresponding to the time coordinate, t of the transient M/M/∞ QM are:

$$\frac{d^2\theta^3}{dt^2} + \Gamma_{ij}^{1(0)}\left(\frac{d\theta^i}{dt}\right)(\frac{d\theta^j}{dt}) = 0, i, j = 1,2,3.$$

Setting $\lambda = 1, \xi = \eta + \varsigma t$, η and ς are any real numbers

$$a = \frac{-1}{\xi^2}, \quad (150)$$

$$b = -\frac{\varsigma}{\xi^2}, \quad (151)$$

$$d_1 = \frac{2}{\xi^3}, \quad (152)$$

$$g = -\frac{2\varsigma}{\xi^3}, \quad (153)$$

$$l = -\frac{\varsigma}{\xi^2}, \quad (154)$$

$$h = \frac{2\varsigma}{\xi^3}, \quad (155)$$

$$r = \frac{2\varsigma^2}{\xi^3}, \quad (156)$$

with

$$L = \frac{(ag-bd_1)}{\Delta} = \frac{(\frac{-1}{\xi^2})(-\frac{2\varsigma}{\xi^3})-(-\frac{\varsigma}{\xi^2})(\frac{2}{\xi^3})}{\Delta} = \frac{4\varsigma}{\Delta\,\xi^5}, \tag{157}$$

$$F = \frac{(ab)}{\Delta} = \frac{\varsigma}{\Delta\xi^4}, \tag{158}$$

$$I = \frac{(-a^2)}{\Delta} = -\frac{1}{\Delta\xi^4}, \tag{159}$$

$$\Gamma_{22}^{3(0)} = \frac{\xi L - 3F - 3\varsigma I}{\xi^4}, \tag{160}$$

$$\Gamma_{23}^{3(\alpha)} = \frac{\varsigma}{\xi^4}(L - 3F - 3\varsigma I), \tag{161}$$

$$\Gamma_{32}^{3(\alpha)} = \frac{\varsigma}{\xi^4}(L\xi - 3F - 3\varsigma I\,), \tag{162}$$

$$\Gamma_{33}^{3(\alpha)} = \frac{\varsigma^2}{\xi^4}(\xi L - 3F - 3\varsigma I). \tag{163}$$

The IMEs of time, for $\lambda = 1, \xi = \eta + \varsigma t$ are given by

$$\frac{d^2t}{dt^2} + \Gamma_{22}^{3(0)}\left(\frac{d\xi}{dt}\right)^2 + \Gamma_{33}^{3(0)}\left(\frac{dt}{dt}\right)^2 + (\Gamma_{23}^{3(0)} + \Gamma_{32}^{3(0)})\left(\frac{d\xi}{dt}\right)\left(\frac{dt}{dt}\right) = 0,$$

or

$$\Gamma_{22}^{3(0)}\left(\frac{d\xi}{dt}\right)^2 + \Gamma_{33}^{3(0)} + (\Gamma_{23}^{3(0)} + \Gamma_{32}^{3(0)})\left(\frac{d\xi}{dt}\right) = 0,$$

implying either

$$\varsigma = 0, \text{or } \xi = \eta = \text{constant}, \tag{164}$$

or

$$\Gamma_{22}^{3(0)}\left(\frac{d\xi}{dt}\right) + \Gamma_{33}^{3(0)} + \Gamma_{23}^{3(0)} + \Gamma_{32}^{3(0)} = 0, \text{or } \eta = \left(\frac{d\xi}{dt}\right) = -\frac{\left(\Gamma_{33}^{3(0)}+\Gamma_{23}^{3(0)}+\Gamma_{32}^{3(0)}\right)}{\Gamma_{22}^{3(0)}} = -\frac{\left(\Gamma_{33}^{3(0)}+\Gamma_{23}^{3(0)}+\Gamma_{32}^{3(0)}\right)}{\Gamma_{22}^{3(0)}} = -\frac{\frac{\varsigma}{\xi^4}(L-3F-3\varsigma I)+\frac{\varsigma}{\xi^4}(L\xi-3F-3\varsigma I\,)+\frac{\varsigma^2}{\xi^4}(\xi L-3F-3\varsigma I)}{\frac{\xi L-3F-3\varsigma I}{\xi^4}} = -\frac{\varsigma(L-3F-3\varsigma I)+\varsigma(L\xi-3F-3\varsigma I\,)+\varsigma^2(\xi L-3F-3\varsigma I)}{\xi L-3F-3\varsigma I} = -\varsigma\frac{(L-3F-3\varsigma I)+(L\xi-3F-3\varsigma I\,)+\varsigma(\xi L-3F-3\varsigma I)}{\xi L-3F-3\varsigma I}. \tag{165}$$

This implies:

$\varsigma = 0, \text{or } \xi = \eta = \text{constant}.$ (c.f., *Eq. (166)*)

or

$$1+\frac{(L-3F-3\varsigma I)+(L\xi-3F-3\varsigma I\,)+\varsigma(\xi L-3F-3\varsigma I)}{\xi L-3F-3\varsigma I} = 0. \tag{166}$$

Following *Eq. (168)*, we have

$$(L - 3F - 3\varsigma I) + (L\xi - 3F - 3\varsigma I\,) + \varsigma(\xi L - 3F - 3\varsigma I) + \xi L - 3F - 3\varsigma I = 0. \tag{167}$$

Consequently, it follows that:

$$\frac{2\varsigma}{\Delta\,\xi^5}[2 - \varsigma\xi + 2\xi] = 0. \tag{168}$$

Eq. (170) holds if and only if

$$\varsigma = 0, \text{or } \xi = \eta = \text{constant.} \tag{169}$$

or

$$\frac{1}{\Delta\, \xi^5} = 0, \text{equivalently } \xi \to \infty, \text{which is equivalent to } \frac{1}{\Delta} = 0. \tag{170}$$

or

$$[2 - \varsigma\xi + 2\xi] = 0. \tag{171}$$

By *Eq. (174)*, we have

$$\frac{\xi d\xi}{1+\xi} = 2dt. \tag{172}$$

It could be verified that *Eq. (175)* has a closed form solution devised by

$$(\xi + 1)e^{\xi} = \zeta e^{-2t}, \text{for some constant } \zeta. \tag{173}$$

The obtained result in *Eq. (176)* is quite new and interesting as it shows that for sufficiently large temporal values, this generates two unexpected values of the Poisson arrival rate, namely , $\xi = -1$ or $\xi \to -\infty$.

At this stage, we need to provide more analysis of the obtained result.

$\frac{1}{\Delta} = 0.$ (*Eq. (173)*)

We have by *Eq. (152)-(158)* and *Eq. (96)*,

$$\Delta = \det\left(\left[g_{ij}\right]\right) = -4\frac{\varsigma^2}{\xi^7}. \tag{174}$$

Combining *Eq. (173)* and *Eq. (177)*, implies either $\xi = 0$ or $\varsigma = \frac{d\xi}{dt} \to \infty$. This means that within the obtaind phase of *Eq. (173)*, either the Poisson arrival rate vanishes, or we will be in a situation where the corresponding velocity of the Poissonian arrival rate is infinite.

7 | The Threshold Theorems for the Potential Function of the Underlying Transient M/M/∞QM

7.1 | The Threshold Theorem for the Potential Function, TTPF

Based on the Preliminary Theorem (PT) , the threshold theorem for the potential function, (*Eq. (44)* of *Theorem 2*) corresponding to each coordinate is devised.

Theorem 5. For the obtained potential function $\Psi(\theta)$(*Eq. (44)* of *Theorem 2*), the following holds

I. $\Psi(\theta)$ is forever increasing in λ.

II. $\Psi(\theta)$ is never increasing in λ.

III. $\Psi(\theta)$ is never decreasing in ξ.

IV. $\Psi(\theta)$ is forever decreasing in ξ for all $\lambda, \xi, t > 0$

V. $\Psi(\theta)$ is forever increasing in t if and only if $\xi > 0$ and temporal values satisfying:

$$(\lambda^{\cdot}\xi - \lambda\xi^{\cdot})e^{\xi t} > (-\lambda\xi\xi^{\cdot}t - \lambda\xi^2 + \lambda^{\cdot}\xi - \lambda\xi^{\cdot}). \tag{175}$$

VI. $\Psi(\theta)$ is never decreasing in t.

Proof:

I. We have

$\partial_1 = \frac{\partial\Psi}{\partial\lambda} = \frac{1}{\xi}(1 - e^{-\xi t}), \partial_2 = \frac{\partial\Psi}{\partial\xi} = \frac{\lambda}{\xi^2}\left((1 + \xi t)e^{-\xi t} - 1\right), \partial_3 = \frac{\partial\Psi}{\partial t} = \frac{1}{\xi^2}\left((\lambda^{\cdot}\xi - \lambda\xi^{\cdot}) + (\lambda\xi\xi^{\cdot} + \lambda\xi^2 + \lambda^{\cdot}\xi - \lambda\xi^{\cdot})e^{-\xi t}\right)$.(c.f., *Eq. (45)*)

It holds that $\partial_1 > 0$ if and only if one of the following statements is true:

$$\frac{1}{\xi} > 0 \text{ and } \left(1 - e^{-\xi t}\right) > 0. \quad \textbf{(176)}$$

$$\frac{1}{\xi} < 0 \text{ and } \left(1 - e^{-\xi t}\right) < 0. \quad \textbf{(177)}$$

The second statement is impossible since it contradicts the fact ξ is positive. So, we have to accept the first statement, *Eq. (179)* implies $e^{\xi t} > 1$, which is always true for all $\xi, t > 0$. This proves I.

As for

II. $\partial_1 < 0$ if and only if one of the following statements is true.

$$\frac{1}{\xi} > 0 \text{ and } \left(1 - e^{-\xi t}\right) < 0. \quad \textbf{(178)}$$

$$\frac{1}{\xi} < 0 \text{ and } \left(1 - e^{-\xi t}\right) > 0. \quad \textbf{(179)}$$

The second statement is impossible since it contradicts the fact ξ is positive. So, we have to accept the first statement, *Eq. (182)* implies $e^{\xi t} < 1$, which is never true for all $\xi, t > 0$. This proves II.

III. $\partial_2 > 0$ if and only if one of the following statements is true.

$$\frac{\lambda}{\xi^2} > 0 \text{ and } \left((1 + \xi t)e^{-\xi t} - 1\right) > 0. \quad \textbf{(180)}$$

$$\frac{\lambda}{\xi^2} < 0 \text{ and } \left((1 + \xi t)e^{-\xi t} - 1\right) < 0. \quad \textbf{(181)}$$

The first statement is impossible since it contradicts the fact ξ is positive. Since, *Eq. (183)* implies $\left((1 + \xi t)e^{-\xi t} - 1\right) > 0$, implying $(1 + \xi t) > e^{\xi t}$, a contradiction always true for all $\xi, t > 0$. Moreover, we have to reject the second statement, since $\frac{\lambda}{\xi^2} < 0$ is never permissible. This proves III.

To show

IV. $\partial_2 < 0$ if and only if one of the following statements is true.

$$\frac{\lambda}{\xi^2} > 0 \text{ and } \left((1 + \xi t)e^{-\xi t} - 1\right) < 0. \quad \textbf{(182)}$$

$$\frac{\lambda}{\xi^2} < 0 \text{ and } \left((1 + \xi t)e^{-\xi t} - 1\right) > 0. \quad \textbf{(183)}$$

The second statement is impossible since it contradicts the fact that $\lambda, \xi > 0$. So, we have to reject *Eq. (186)*. The first statement, *Eq. (186)* implies $\left((1 + \xi t)e^{-\xi t} - 1\right) < 0$, implying $(1 + \xi t) > e^{\xi t}$, which is always true for all $\xi, t > 0$. Also, $\frac{\lambda}{\xi^2} > 0$ holds for all $\lambda, \xi > 0$. This proves IV.

V. We have $\partial_3 = \frac{\partial\Psi}{\partial t} = \frac{1}{\xi^2}\left((\lambda^{\cdot}\xi - \lambda\xi^{\cdot}) + (\lambda\xi\xi^{\cdot}t + \lambda\xi^2 - \lambda^{\cdot}\xi + \lambda\xi^{\cdot})e^{-\xi t}\right)$. So, $\partial_3 > 0$ if and only if one of the following statements is true.

$$\frac{1}{\xi^2} > 0 \text{ and } \left((\lambda^{\cdot}\xi - \lambda\xi^{\cdot}) + (\lambda\xi\xi^{\cdot}t + \lambda\xi^2 - \lambda^{\cdot}\xi + \lambda\xi^{\cdot})e^{-\xi t}\right) > 0. \quad \textbf{(184)}$$

$$\frac{1}{\xi^2} < 0 \text{ and } \left((\lambda^{\cdot}\xi - \lambda\xi^{\cdot}) + (\lambda\xi\xi^{\cdot}t + \lambda\xi^2 - \lambda^{\cdot}\xi + \lambda\xi^{\cdot})e^{-\xi t}\right) < 0. \quad \textbf{(185)}$$

We have to reject *Eq. (188)* since $\frac{1}{\xi^2} < 0$ is non- permissible. As for *Eq. (187)*, $\frac{1}{\xi^2} > 0$ holds. Furthermore,

$\left((\lambda^{\cdot}\xi - \lambda\xi^{\cdot}) + (\lambda\xi\xi^{\cdot}t + \lambda\xi^2 - \lambda^{\cdot}\xi + \lambda\xi^{\cdot})e^{-\xi t}\right) > 0$ implies

$$(\lambda^{\cdot}\xi - \lambda\xi^{\cdot})e^{\xi t} = (\lambda^{\cdot}\xi - \lambda\xi^{\cdot})\left(1 + \xi t + \frac{(\xi t)^2}{2} + \cdots \ldots.\right) = (\lambda^{\cdot}\xi - \lambda\xi^{\cdot} - \lambda\xi^{\cdot}\xi t + \cdots.) > (-\lambda\xi\xi^{\cdot}t - \lambda\xi^2 + \lambda^{\cdot}\xi - \lambda\xi^{\cdot}). \quad (186)$$

Hence, $(\lambda^{\cdot}\xi - \lambda\xi^{\cdot})e^{\xi t} > (-\lambda\xi\xi^{\cdot}t - \lambda\xi^2 + \lambda^{\cdot}\xi - \lambda\xi^{\cdot})$ holds. This proves V.

VI. We have $\partial_3 = \frac{\partial\Psi}{\partial t} = \frac{1}{\xi^2}\left((\lambda^{\cdot}\xi - \lambda\xi^{\cdot}) + (\lambda\xi\xi^{\cdot}t + \lambda\xi^2 - \lambda^{\cdot}\xi + \lambda\xi^{\cdot})e^{-\xi t}\right)$. So, $\partial_3 < 0$ if and only if one of the following.

statement is true:

$$\frac{1}{\xi^2} > 0 \text{ and } \left((\lambda^{\cdot}\xi - \lambda\xi^{\cdot}) + (\lambda\xi\xi^{\cdot}t + \lambda\xi^2 - \lambda^{\cdot}\xi + \lambda\xi^{\cdot})e^{-\xi t}\right) < 0. \quad (187)$$

$$\frac{1}{\xi^2} < 0 \text{ and } \left((\lambda^{\cdot}\xi - \lambda\xi^{\cdot}) + (\lambda\xi\xi^{\cdot}t + \lambda\xi^2 - \lambda^{\cdot}\xi + \lambda\xi^{\cdot})e^{-\xi t}\right) > 0. \quad (188)$$

We must reject *Eq. (191)*. *Eq. (190)* is true since $\frac{1}{\xi^2} > 0$ holds for all $\xi > 0$. Moreover, we have

$$\left((\lambda^{\cdot}\xi - \lambda\xi^{\cdot}) + (\lambda\xi\xi^{\cdot}t + \lambda\xi^2 - \lambda^{\cdot}\xi + \lambda\xi^{\cdot})e^{-\xi t}\right) < 0. \quad (189)$$

Let $\xi = 1 = \lambda, t = 0$. This provides a counter example generating the inequality,

$$1 < 0. \quad (190)$$

A contradiction. Therefore, VI holds.

In what follows, $\text{PAR} = \xi = \text{Poissopn Arrival Rate}, \text{AR} = \lambda = \text{Arrival Rate}, \text{PF} = \text{Potential Function} = \Psi(\theta)$

7.2.1 | Numerical experiment one

$\Psi(\theta) = \lambda(1 - e^{-1})$.

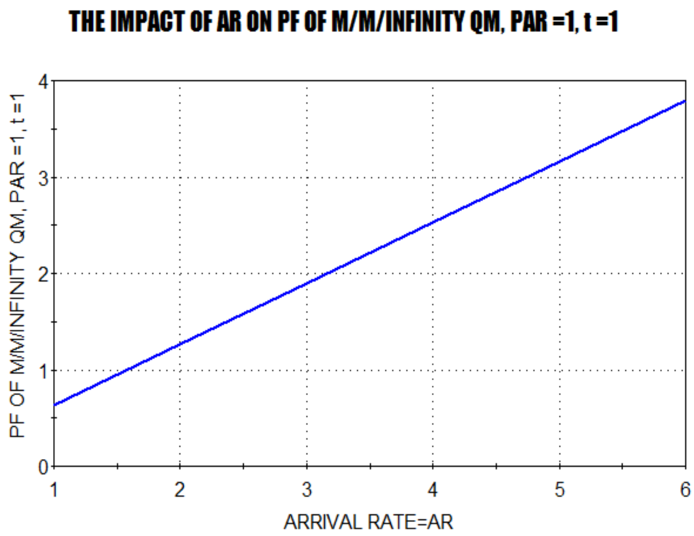

Fig. 5. Visualization of the impact of arrival rate on the potential function.

The numerical observation in figure 5matches the analytic result of *Theorem 5*.

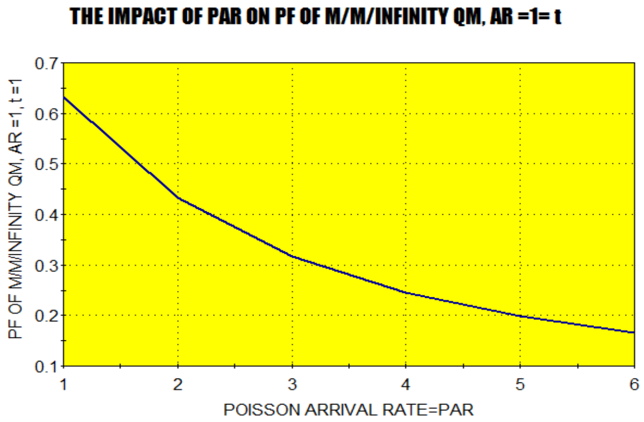

Fig. 6. Visualization of the impact of Poisson arrival rate on the potential function.

Fig. 6 provides an evidence of the forever decreasability phase of the PF in the Poisson arrival rate, which agrees with the analytic findings of TTPF.

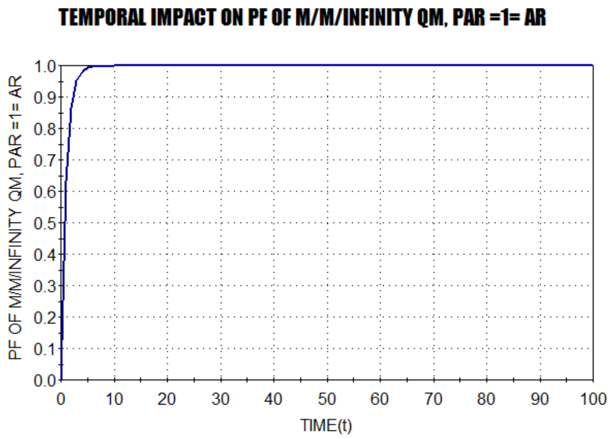

Fig. 7. Visualization of the temporal impact of arrival rate on the potential function.

As observed from *Fig. 7*, the potential function, PF is forever increasing in time, which agrees with the analytic results.

8 | Some Algebraic Properties of the Potential Function, $\boldsymbol{\Psi(\theta) = \frac{\lambda}{\xi}(1 - e^{-\xi t})}$

Theorem 6. The three-dimensional potential function $\Psi(\theta) = \frac{\lambda}{\xi}(1 - e^{-\xi t})$(*Eq. (44)*) is generally not well-defined.

Proof: let λ_i, ξ_i, t_i , $i = 1,2$ be such that $\lambda_1 \neq \lambda_2$, $\xi_1 \neq \xi_2$, $t_1 \neq t_2$. Let $\Psi(\lambda_1, \xi_1, t_1) = \Psi(\lambda_2, \xi_2, t_2)$. This implies

$$\frac{\lambda_1}{\xi_1}\left(1 - e^{-\xi_1 t_1}\right) = \frac{\lambda_2}{\xi_2}\left(1 - e^{-\xi_2 t_2}\right). \quad (191)$$

Eq. (194) directly implies:

$$\left[\frac{\lambda_1}{\xi_1} - \frac{\lambda_2}{\xi_2}\right] + \left[\frac{\lambda_2}{\xi_2} e^{-\xi_2 t_2} - \frac{\lambda_1}{\xi_1} e^{-\xi_1 t_1}\right] = 0. \quad (192)$$

So, *Eq. (195)* holds if and only if:

$$\frac{\lambda_2}{\xi_2}\sum_{m=1}^{\infty}\frac{(-\xi_2 t_2)^m}{m!}-\frac{\lambda_1}{\xi_1}\sum_{m=1}^{\infty}\frac{(-\xi_1 t_1)^m}{m!}=0. \quad \textbf{(193)}$$

Clearly *Eq. (196)* is satisfied if and only if:

$$\frac{(-1)^m}{m!}[\lambda_2(\xi_2)^{m-1}(t_2)^m-\lambda_1(\xi_1)^{m-1}(t_1)^m], m=1,2,.... \quad \textbf{(194)}$$

It can be verified that *Eq. (197)* will generate the following sets of equations:

$$m=1 \Longrightarrow \lambda_2 t_2=\lambda_1 t_1. \quad \textbf{(195)}$$

$$m=2 \Longrightarrow [\lambda_2\xi_2(t_2)^2-\lambda_1\xi_1(t_1)^2\ ]=0. \quad \textbf{(196)}$$

Engaging *Eq. (198)* and *Eq. (199)* yields

$\lambda_1 t_1(\xi_2 t_2-\xi_1 t_1)=0 \implies \lambda_1 t_1=0$ (which implies λ_1 t_1=0, or λ_1=0 (contradiction), or t_1=0).If t_1=0, then this implies by *Eq. (198)* that either λ_2=0 or t_2=0).

In brief, this implies that either all the temporal values are set to zero, which is a contradiction. Moreover, the other possibility is to allow the arrival rate values to be zero, another contradiction .To this end, we have

$$\xi_2 t_2=\xi_1 t_1. \quad \textbf{(197)}$$

The reader can check that after some manipulation,

$$m=3 \Longrightarrow \xi_2 t_2=\xi_1 t_1 \quad \text{or } \xi_2 t_2+\xi_1 t_1=0 \text{ (contradiction)}. \quad \textbf{(198)}$$

Based on the analytic results, we can have $\lambda_1 \neq \lambda_2$, $\xi_1 \neq \xi_2, t_1 \neq t_2$ and $\Psi(\lambda_1,\xi_1,t_1)=\Psi(\lambda_2,\xi_2,t_2)$ will hold. This means that Ψ is not well-defined.

Several emerging important special cases of *Theorem 6* are obtained in the following theorems.

Theorem 7. For constant values of λ,ξ, the three-dimensional potential function $\Psi(\theta)=\frac{\lambda}{\xi}(1-e^{-\xi t})$(c.f., *Eq. (43)*) satisfies the following:

I. Ψ is well-defined.

II. Ψ is onto.

III. Ψ is One − to − One.

IV. Ψ has a unique inverse, Ψ^{-1} given by

$$\Psi^{-1}(\lambda,\xi,t)=-\frac{1}{\xi}\ln\left(1-\frac{t\xi}{\lambda}\right). \quad \textbf{(199)}$$

Proof: for constant values of λ,ξ, define $t_i, i=1,2$ be such that $t_1 \neq t_2$. Thus, $\Psi(\lambda,\xi,t_1)=\Psi(\lambda,\xi,t_2)$ implies:

$$\frac{\lambda}{\xi}(1-e^{-\xi t_1})=\frac{\lambda}{\xi}(1-e^{-\xi t_2}). \quad \textbf{(200)}$$

Hence, it follows that:

$$\frac{\lambda}{\xi}(e^{-\xi t_1}-e^{-\xi t_2})=0 \iff \frac{\lambda}{\xi}\text{(contradiction) or } e^{-\xi t_1}=e^{-\xi t_2}\text{, equivalently } t_1=t_2. \quad \textbf{(201)}$$

This proves (1).

For every arbitrary $\frac{\lambda}{\xi}(1-e^{-\xi t})$, for constant values of λ,ξ there exist a unique triple (λ,ξ,t) such that the that th representation $\Psi(\theta)=\frac{\lambda}{\xi}(1-e^{-\xi t})$ exists. Therefore, Ψ is onto.

It suffices to show that for constant values of λ, ξ, it holds that:

$$\Psi(\lambda, \xi, t_1) = \Psi(\lambda, \xi, t_2) \iff t_1 = t_2. \quad \textbf{(202)}$$

$\Longrightarrow$: follows by (1).

$\Longleftarrow$: $t_1 = t_2$ implies $\Psi(\lambda, \xi, t_1) = \Psi(\lambda, \xi, t_2)$. The proof is immediate.

Combining (2) and (3), Ψ is bijective with a unique inverse, namely Ψ^{-1}. To obtain Ψ^{-1}, let

$$\Psi(\theta) = \frac{\lambda}{\xi}\left(1 - e^{-\xi t}\right) = y. \quad \textbf{(203)}$$

Hence, it follows that

$$\left(1 - e^{-\xi t}\right) = \frac{\xi y}{\lambda} \Longrightarrow e^{-\xi t} = 1 - \frac{\xi y}{\lambda}, \text{equivalently}, t = -\frac{1}{\xi}\ln\left(1 - \frac{\xi y}{\lambda}\right). \quad \textbf{(204)}$$

Following *Eq. (207)*, we get

$\Psi^{-1}(t) = -\frac{1}{\xi}\ln\left(1 - \frac{\xi t}{\lambda}\right)$. (c.f., *Eq. (202)*)

Theorem 8. For constant $\xi, t \to \infty$ the potential function $\Psi_{\infty,\lambda}(\theta) = \frac{\lambda}{\xi}$ (c.f., (3.23)) satisfies the following:

I. $\Psi_{\infty,\lambda}(\theta)$ is well-defined.

II. $\Psi_{\infty,\lambda}(\theta)$ is onto.

III. $\Psi_{\infty,\lambda}(\theta)$ is One − to − One.

IV. $\Psi_{\infty,\lambda}(\theta)$ has a unique inverse, $\Psi^{-1}_{\infty,\lambda}$ given by

$$\Psi^{-1}_{\infty,\lambda}(\lambda) = \lambda\xi. \quad \textbf{(205)}$$

Proof: following a similar approach as in *Theorem 7*, the proofs are straightforward.

Theorem 9. For constant $\lambda, t \to \infty$ the potential function $\Psi_{\infty,\xi}(\theta) = \frac{\lambda}{\xi}$ (*Eq. (43)*) satisfies the following:

I. $\Psi_{\infty,\xi}$ is well-defined.

II. $\Psi_{\infty,\xi}$ is onto.

III. $\Psi_{\infty,\xi}$ is One − to − One.

IV. $\Psi_{\infty,\xi}$ has a unique inverse, $\Psi^{-1}_{\infty,\xi}$ given by

$$\Psi^{-1}_{\infty,\xi}(\xi) = \frac{\lambda}{\xi}. \quad \textbf{(206)}$$

Proof: following a similar approach as in *Theorem 7*, the proofs are straightforward.

Theorem 10. For constant λ, the potential function $\Psi_{\xi,t}(\theta) = \frac{\lambda}{\xi}(1 - e^{-\xi t})$ (c.f., *Eq. (43)*) satisfies the following:

I. $\Psi_{\xi,t}(\theta)$ is well-defined if and only if the temporal values are constant.

II. $\Psi_{\xi,t}(\theta)$ is onto.

III. $\Psi_{\xi,t}(\theta)$ is not in general One − to − One.

IV. $\Psi_{\xi,t}(\theta)$ has a family of families of inverses, $\Psi^{-1}_{\xi,t}$ and they are existing uniquely if and only if the temporal values are constant.

In this case,

$\Psi_{\xi,t}^{-1}(\xi) = \xi.$

Such that, ξ satisfies:

$$\lambda\left[\sum_{m=1}^{\infty}\frac{(-\xi)^{m-2}(t)^{m-1}}{m!}\right]-1=0. \quad (207)$$

Proof: let ξ_i, t_i , $i = 1,2$ be such that $\xi_1 \neq \xi_2, t_1 \neq t_2$. Let $\Psi(\lambda,\xi_1,t_1) = \Psi(\lambda,\xi_2,t_2)$. This implies

$$\frac{\lambda}{\xi_1}\left(1-e^{-\xi_1 t_1}\right)=\frac{\lambda}{\xi_2}\left(1-e^{-\xi_2 t_2}\right). \quad (208)$$

Eq. (201) directly implies:

$$\lambda(\left[\frac{1}{\xi_1}-\frac{1}{\xi_2}\right]+\left[\frac{1}{\xi_2}e^{-\xi_2 t_2}-\frac{1}{\xi_1}e^{-\xi_1 t_1}\right])=0. \quad (209)$$

For the Necessity, if the temporal values are constant, then $t_1 = t_2$ holds, which implies by (*Eq. (212)*, that $\xi_1 = \xi_2$, since $\lambda = 0$ is non permissible by the hypothesis. Therefore, $\Psi_{\xi,t}(\theta)$ is well-defined.

For the sufficiency, let $\Psi_{\xi,t}(\theta)$ be well-defined, then it is never permissible that:

$$\xi_1 \neq \xi_2, t_1 \neq t_2 \text{ to imply } \Psi(\lambda,\xi_1,t_1)=\Psi(\lambda,\xi_2,t_2). \quad (210)$$

So, $t_1 = t_2$ and $\xi_1 = \xi_2$ hold. Assume that there is some t_m such that $\xi_1 \neq \xi_2, t_1 \neq t_m$ to imply $\Psi(\lambda,\xi_1,t_1) = \Psi(\lambda,\xi_2,t_m)$. Then, we get a contradiction to the hypothesis of well-defindness of $\Psi_{\xi,t}(\theta)$ .Thus, $\xi_1 = \xi_2, t_1 = t_m = t_2$. Therefore, the temporal values should be constant. This completes the proof of (1).

The proof of (2) is straightforward.

It suffices to show that for constant values of λ, we have for any arbitrary ξ_1, ξ_2, t_1, t_2

$$\Psi(\lambda,\xi,t_1)=\Psi(\lambda,\xi,t_2) \Leftrightarrow \xi_1=\xi_2 \text{ and } t_1=t_2. \quad (211)$$

Following the proof of (1), the necessity condition holds if the temporal values are constant. Furthermore, the sufficiency condition implies that $\Psi(\lambda,\xi,t_1) = \Psi(\lambda,\xi,t_2) \Rightarrow \xi_1 = \xi_2$, which holds if and only if $t_1 = t_2$. Repeating implies the same procedure iteratively would imply $t_1 = t_2 = t_3 = t_4 = \cdots$. This provides an evidence that all the temporal values are constant.

The first part of the proof of (4) is clear. Now, assume that all the temporal values are constant. Then, of course, there should be a uniquely defined and determined inverse, namely, $\Psi_{\xi,t}^{-1}$.Let $\Psi_{\xi,t}(\theta) = \frac{\lambda}{\xi}(1 - e^{-\xi t}) = z$, then it holds that

$$z=\frac{\lambda}{\xi}\left(1-\left[\sum_{m=0}^{\infty}\frac{(-t\xi)^m}{m!}\right]\right) \Rightarrow \frac{z}{\lambda}=\left[\sum_{m=1}^{\infty}\frac{(-1)^{m-2}(\xi)^{m-1}(t)^{m-1}}{m!}\right]. \quad (212)$$

Therefore, the inverse potential function, $\Psi_{\xi,t}^{-1}$ is devised by

$$\Psi_{\xi,t}^{-1} = \text{ the roots of the equation } \lambda\left[\sum_{m=1}^{\infty}\frac{(-\xi)^{m-2}(t)^{m-1}}{m!}\right]-1=0. \text{(c.f., } \textit{Eq. (210)})$$

Theorem 8.6 For constant ξ, the potential function $\Psi_{\lambda,t}(\theta) = \frac{\lambda}{\xi}(1 - e^{-\xi t})$(c.f., (3.23)) satisfies the following:

I. $\Psi_{\lambda,t}(\theta)$ is well-defined if and only if the temporal values are constant.

II. $\Psi_{\lambda,t}(\theta)$is onto

III. $\Psi_{\lambda,t}(\theta)$ is not in general One − to − One.

IV. $\Psi_{\lambda,t}(\theta)$ has a family of families of inverses, $\Psi_{\lambda,t}^{-1}$ and it is existing uniquely if and only if the temporal values are constant.

In this case,

$$\Psi_{\lambda,t}^{-1}(\lambda) = \frac{\lambda\xi}{(1 - e^{-\xi t})}. \tag{213}$$

Proof: following the same technique as in *Theorem 10*, the proofs can be easily done.

The following section provides the threshold theorems of the derived inverse of the potential functions (*Eq. (202)*, *Eq. (208)*, *Eq. (209)*, *Eq. (216)*.

9 | The Threshold Theorems of the Derived Inverses of Poten-Tial Function, IPFs

Theorem 12. For the derived inverse, $\Psi^{-1}(t) = -\frac{1}{\xi}\ln\left(1 - \frac{\xi t}{\lambda}\right)$ (*Eq. (202)*), we have

I. $\Psi^{-1}(t)$ is forever increasing in t if and only if:

$$t < \frac{\lambda}{\xi}. \tag{214}$$

II. $\Psi^{-1}(t)$ is never decreasing in t.

Proof:

I. We have

$$\frac{\partial\Psi^{-1}(t)}{\partial t} = \frac{-\frac{1}{\xi}\left(\frac{\xi}{\lambda}\right)}{\left(1 - \frac{\xi t}{\lambda}\right)} = \frac{\frac{1}{\lambda}}{\left(1 - \frac{\xi t}{\lambda}\right)} = \frac{1}{(\lambda - \xi t)}. \tag{215}$$

By the preliminary theorem (PT)(c.f., *Eq. (15)*), Ψ^{-1} is forever increasing if and if $\frac{\partial\Psi^{-1}(t)}{\partial t} > 0$. Following *Eq. (215)*, this holds if and only if $(\lambda - \xi t) > 0$. Surely, it is implied that i) holds.

Engaging the same approach, $\frac{\partial\Psi^{-1}(t)}{\partial t} < 0$ if and only if $> \frac{\lambda}{\xi}$, but this implies $\frac{\xi t}{\lambda} > 1$, which violates the continuity of $\Psi^{-1}(t)$. Hence, ii) follows.

Theorem 13. For the obtained inverse, $\Psi_{\infty,\lambda}^{-1}(\lambda) = \lambda\xi$ (c.f., *Eq. (208)*), we have

I. $\Psi_{\infty,\lambda}^{-1}(\lambda)$ is forever increasing in λ.

II. $\Psi_{\infty,\lambda}^{-1}(\lambda)$is never decreasing in λ.

Proof

I. We have

$$\frac{\partial\Psi_{\infty,\lambda}^{-1}(\lambda)}{\partial\lambda} = \xi. \tag{216}$$

By the PT (*Eq. (15)*), $\Psi_{\infty,\lambda}^{-1}(\lambda)$ is forever increasing if and if $\frac{\partial\Psi_{\infty,\lambda}^{-1}(\lambda)}{\partial\lambda} > 0$. By *Eq. (216)*, this is satisfied generally since ξ is always positive.

The proof of II follows since ξ is never negative.

Theorem 14. For the obtained inverse, $\Psi_{\infty,\xi}^{-1}(\xi) = \frac{\lambda}{\xi}$ (*Eq. (209)*), we have

I. $\Psi^{-1}_{\infty,\xi}(\xi)$ is never increasing in ξ.

II. $\Psi^{-1}_{\infty,\xi}(\xi)$ is forever decreasing in ξ.

Proof

I. We have

$$\frac{\partial\Psi^{-1}_{\infty,\xi}(\xi)}{\partial\xi}=-\frac{\lambda}{\xi^2}. \tag{217}$$

By the PT (*Eq. (15)*), $\Psi^{-1}_{\infty,\xi}(\xi)$ is forever increasing if and if $\frac{\partial\Psi^{-1}_{\infty,\xi}(\xi)}{\partial\xi}>0$. Following *Eq. (217)*, this never holds since ξ and λ are always positive. This proves i).

The proof of II is immediate.

Theorem 15. For the obtained inverse, $\Psi^{-1}_{\lambda,t}(\lambda)=\frac{\lambda\xi}{(1-e^{-\xi t})}$ (*Eq. (216)*), we have

I. $\Psi^{-1}_{\lambda,t}(\lambda)$ is never increasing in λ.

II. $\Psi^{-1}_{\lambda,t}(\lambda)$ is never decreasing in λ.

III. $\Psi^{-1}_{\lambda,t}(\lambda)$ is forever increasing in t if and only if

$$t>\frac{1}{\xi}\ln\left(1+\frac{\lambda\xi}{\lambda^{\cdot}}\right). \tag{218}$$

IV. $\Psi^{-1}_{\lambda,t}(\lambda)$ is forever decreasing in t if and only if

$$t<\frac{1}{\xi}ln\left(1+\frac{\lambda\xi}{\lambda^{\cdot}}\right). \tag{219}$$

Proof

I. We have

$$\frac{\partial\Psi^{-1}_{\lambda,t}(\lambda)}{\partial\lambda}=\frac{\xi}{(1-e^{-\xi t})}=\frac{\xi e^{\xi t}}{(e^{\xi t}-1)}>0(\text{since }\xi\text{ is always paosive}, t>0). \tag{220}$$

By the PT (*Eq. (15)*), *Eq. (220)*, it follows that $\Psi^{-1}_{\lambda,t}(\lambda)$ is forever increasing in λ .This proves I.

The proof of ii) is immediate.

$$\frac{\partial\Psi^{-1}_{\lambda,t}(\lambda)}{\partial t}=\frac{\xi(\lambda^{\cdot}\left(1-e^{-\xi t}\right)+\lambda\xi e^{-\xi t})}{(1-e^{-\xi t})^2}. \tag{221}$$

By the PT ((*Eq. (15)*), it follows that $\Psi^{-1}_{\lambda,t}(\lambda)$ is forever increasing in t if and only if:

$$\lambda^{\cdot}\left(1-e^{-\xi t}\right)+\lambda e^{-\xi t}(\xi)>0,\text{ equivalently},\lambda^{\cdot}>\lambda\frac{\xi e^{-\xi t}}{(1-e^{-\xi t})}=\lambda\frac{\xi}{(e^{\xi t}-1)}. \tag{222}$$

Eq. (222) could be re-written in the form:

$$\left(e^{\xi t}-1\right)>\frac{\lambda\xi}{\lambda^{\cdot}},\text{ equivalently}, t>\frac{1}{\xi}\ln\left(1+\frac{\lambda\xi}{\lambda^{\cdot}}\right).\quad\text{(c.f., }Eq.\ (218)) \tag{223}$$

Hence, III is done.

As for IV following the PT (*Eq. (15)*), *Eq. (221)* it follows that $\Psi^{-1}_{\lambda,t}(\lambda)$ is forever decreasing in t if and only if

$$\lambda^{\cdot}\left(1-e^{-\xi t}\right)+\lambda e^{-\xi t}(\xi)<0, \text{equivalently}, \lambda^{\cdot}<\lambda\frac{\xi e^{-\xi t}}{(1-e^{-\xi t})}=\lambda\frac{\xi}{(e^{\xi t}-1)}. \quad (224)$$

Eq. (223) could be re-written in the form:

$\left(e^{\xi t}-1\right)<\frac{\lambda\xi}{\lambda^{\cdot}}$, equivalently, $t<\frac{1}{\xi}\ln(1+\frac{\lambda\xi}{\lambda^{\cdot}})$. (*Eq. (219)*)

10 | Numerical Experiments on the Threshold Theorems of the Derived Inverses of Potential Function

10.1 | Numerical Experiment on Theorem 12

We have, the inverse of the potential function, IPF = $\Psi^{-1}(t)=-\frac{1}{\xi}\ln\left(1-\frac{\xi t}{\lambda}\right)$(*Eq. (202)*). Let the arrival rate, AR= $\lambda=0.1$, the Poissonian arrival rate = PAR = $\xi=0.2$, then $\frac{\lambda}{\xi}=0.2$. If $\frac{\xi t}{\lambda}>1$,or equivalently, $t>0.5$, then $\Psi^{-1}(t)\rightarrow\infty$.

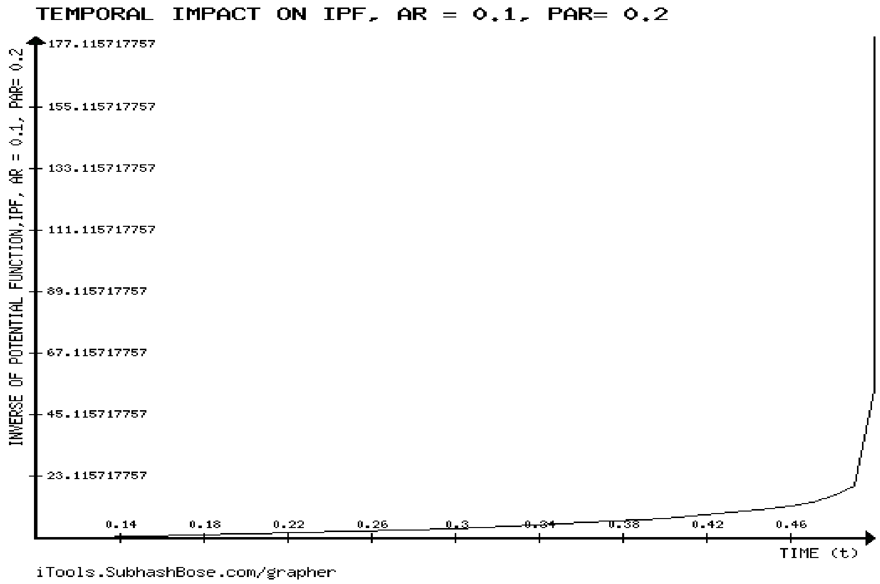

Fig. 8. Temporal impact of arrival rate on the inverse potential function.

As we can see from *Fig. 8* that the inverse of potential function, IPF is forever increasing for all the temporal values less than the threshold = $\frac{\lambda}{\xi}$, where both λ,ξ are constants. These experimental results agree with the analytic findings of *Theorem 12*.

12.2 | Numerical Experiment on Theorem 12

We have, the inverse of the potential function, IPF = $\Psi^{-1}_{\infty,\lambda}(\lambda)=\lambda\xi$ (c.f., *Eq. (208)*). $PAR=\xi=0.2$.

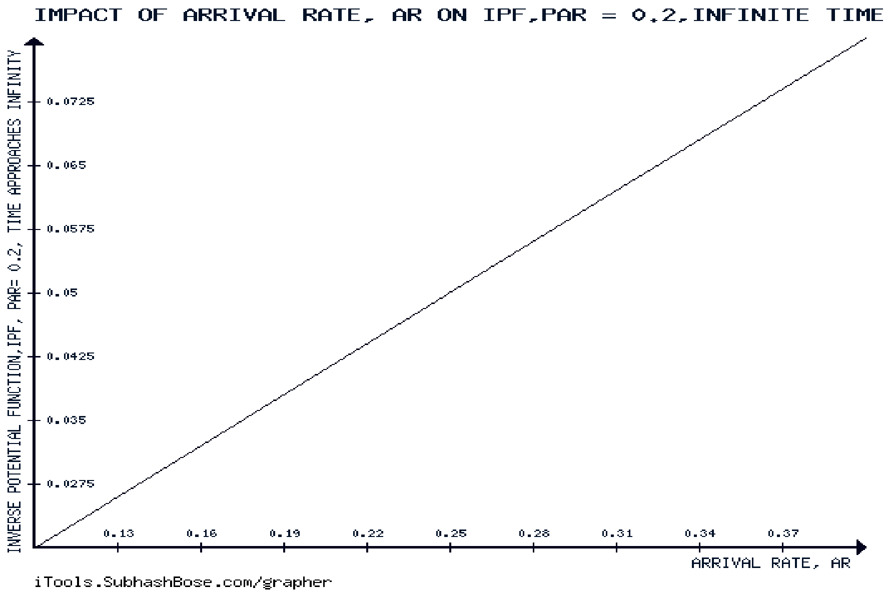

Fig. 9. Visualization of the impact of arrival rate on the potential function, for infinite time.

Fig. 9 shows that the inverse of potential function, IPF is forever increasing with respect to the arrival rate, for infinite time and constant Poissonian arrival rate. This agrees with the analytic results of *Theorem 16*.

10.3 | Numerical Experiment on Theorem 14

We have, the inverse of the potential function, IPF = $\Psi^{-1}_{\infty,\xi}(\xi) = \frac{\lambda}{\xi}$ *(Eq. (2098))*. AR = $\lambda = 0.1$.

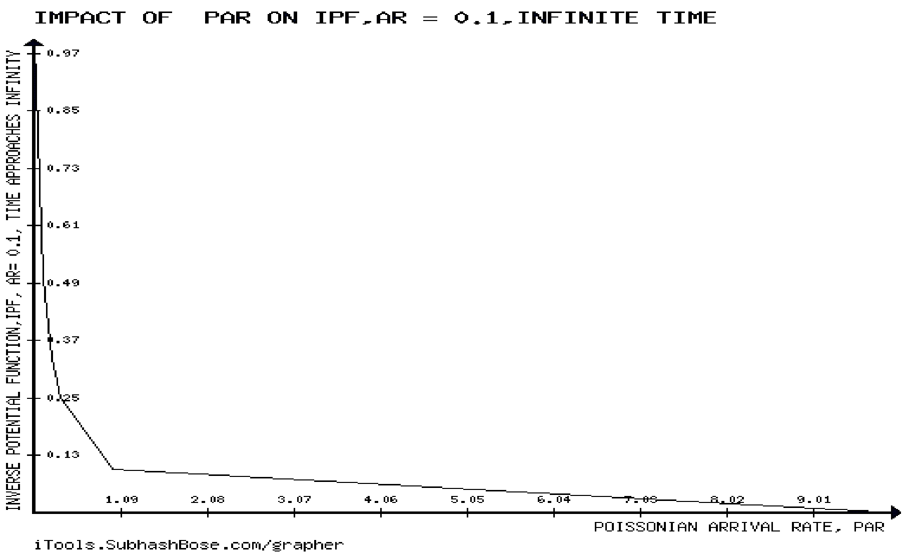

Fig.10. Visualization of the impact of Poisson arrival rate on the inverse potential function, for infinite time.

Fig. 10 shows that IPF is forever decreasing in PAR = Poissonian arrival rate, for constant arrival rate and infinite time. This agrees with the analytic findings of *Theorem 17*. It is also clear that,

$$\Psi^{-1}_{\infty,\xi}(\xi) = \frac{\lambda}{\xi} \to 0 \text{ as } \xi \to \infty.$$

10.4 | Numerical Experiment on Theorem 15

We have, the inverse of the potential function, IPF = $\Psi^{-1}_{\lambda,t}(\lambda) = \frac{\lambda\xi}{(1-e^{-\xi t})}$ *(Eq. (216))*. $PAR = \xi = 0.2$.

Part one: let $t = 5$.

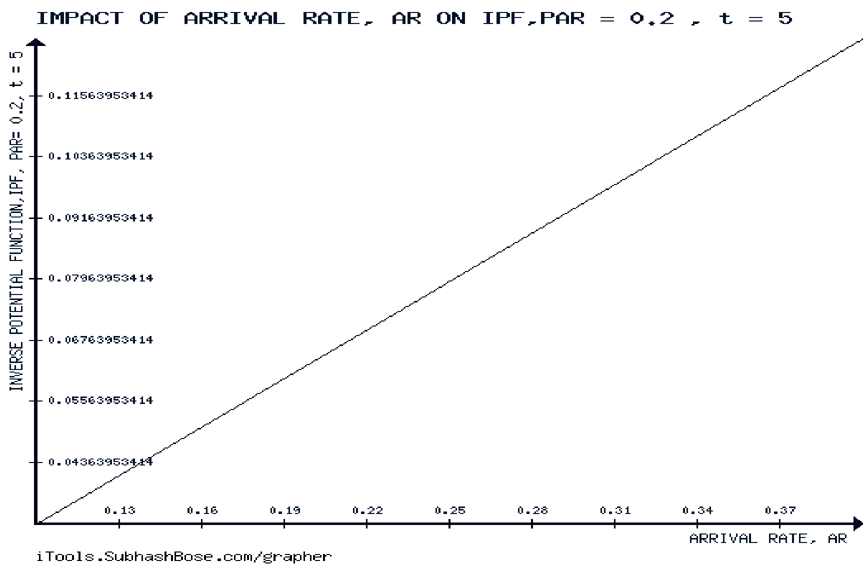

Fig. 11. The impact of arrival rate on the inverse potential function, for $t = 5$, $\textbf{Poisson arrival rate} = 0.2$.

As observed from *Fig. 11*, $\Psi_{\lambda,t}^{-1}(\lambda)$ is forever increasing in λ. This shows that both analytic and numerical results match .

Part two: let $\xi = 0.2, \lambda = e^{(0.1)t}$, we have the threshold to be:

$$\frac{1}{\xi}\ln\left(1+\frac{\lambda\xi}{\lambda\cdot}\right) = \frac{1}{0.2}\ln\left(1+\frac{0.2e^{(0.1)t}}{0.1e^{(0.1)t}}\right) = \frac{1}{0.2}\ln\left(1+\frac{0.2e^{(0.1)t}}{0.1e^{(0.1)t}}\right) = \frac{1}{0.2}\ln(3) = 5.493061443. \quad \textbf{(225)}$$

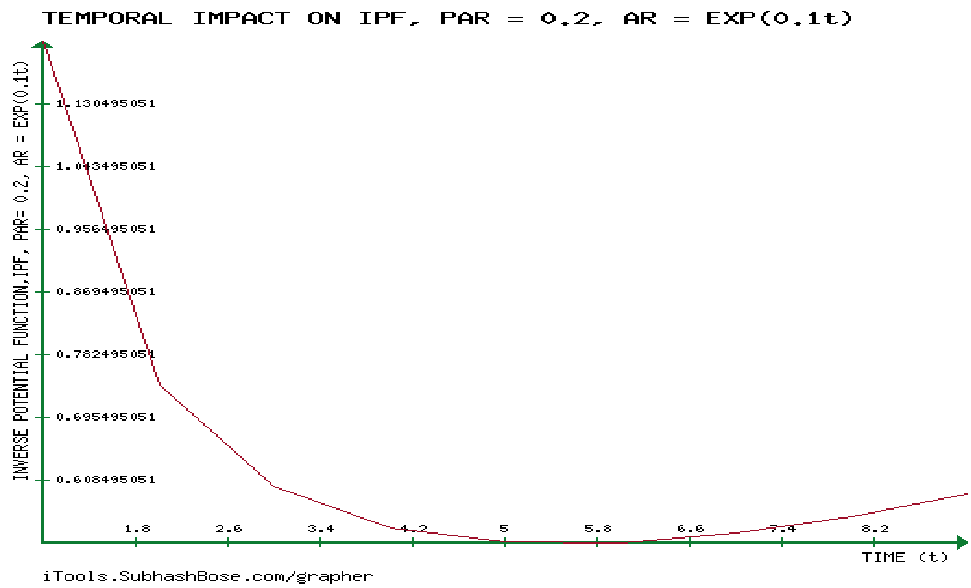

Fig. 12. The impact of arrival rate $e^{0.1t}$, on the inverse potential function, $\textbf{Poisson arrival rate} = 0.2$.

As observed from *Fig. 12*, $\Psi_{\lambda,t}^{-1}(\lambda) = \frac{\lambda\xi}{(1-e^{-\xi t})}$ is forever decreasing whenever time is less than the threshold and it starts to increase when time is greater than the threshold.

11 | The $\alpha-$ Gaussian Curvature, $K_{\infty}^{(\alpha)}$ of Eq. (14) as Time Approaches Infinity

Theorem 18. The $\alpha -$ Gaussian Curvature, $K_{\infty}^{(\alpha)}$ as time approaches infinity is devised by

$$K_{\infty}^{(\alpha)} = \left[\left(\frac{(1-\alpha)\lambda^{\cdot}\xi^{\cdot}\xi^{4}(2\lambda\xi^{\cdot}-\lambda^{\cdot}\xi+2\lambda-\xi)(4\xi^{\cdot 2}-\xi\xi^{\cdot\cdot})}{(\lambda^{\cdot\cdot}\xi^{2}-\lambda\xi\xi^{\cdot\cdot}-4\lambda^{\cdot}\xi\xi^{\cdot}+4\lambda\xi^{\cdot 2})^{3}}\right)\right]. \quad (226)$$

Proof: we have

$$R_{ijkl}^{(\alpha)} = \left[\left(\partial_j\Gamma_{ik}^{s(\alpha)} - \partial_i\Gamma_{jk}^{s(\alpha)}\right)g_{sl} + \left(\Gamma_{j\beta,l}^{(\alpha)}\Gamma_{ik}^{\beta(\alpha)} - \Gamma_{i\beta,l}^{(\alpha)}\Gamma_{jk}^{\beta(\alpha)}\right)\right],$$

where $\Gamma_{ij}^{k(\alpha)} = \Gamma_{ij,s}^{(\alpha)}g^{sk}$

Therefore,

$$R_{1212}^{(\alpha)} = \left[\left(\partial_2\Gamma_{11}^{s(\alpha)} - \partial_1\Gamma_{21}^{s(\alpha)}\right)g_{sl} + \left(\Gamma_{j\beta,l}^{(\alpha)}\Gamma_{11}^{\beta(\alpha)} - \Gamma_{i\beta,l}^{(\alpha)}\Gamma_{21}^{\beta(\alpha)}\right)\right]$$

$$\begin{bmatrix} \left(\frac{\partial}{\partial\xi}[\Gamma_{11}^{1(\alpha)}+\Gamma_{11}^{2(\alpha)}+\Gamma_{11}^{3(\alpha)}] - \frac{\partial}{\partial\lambda}[\Gamma_{21}^{1(\alpha)}+\Gamma_{21}^{2(\alpha)}+\Gamma_{21}^{3(\alpha)}]\right)(g_{12}+g_{22}+g_{32}) \\ +\left([\Gamma_{21,2}^{(\alpha)}\Gamma_{11}^{1(\alpha)}+\Gamma_{22,2}^{(\alpha)}\Gamma_{11}^{2(\alpha)}+\Gamma_{23,2}^{(\alpha)}\Gamma_{11}^{3(\alpha)}] - [\Gamma_{11,2}^{(\alpha)}\Gamma_{21}^{1(\alpha)}+\Gamma_{12,2}^{(\alpha)}\Gamma_{21}^{2(\alpha)}+\Gamma_{13,2}^{(\alpha)}\Gamma_{21}^{3(\alpha)}]\right) \end{bmatrix} \quad (227)$$

$$\begin{bmatrix} \left(-\frac{(1-\alpha)}{2\xi^3}\frac{\partial}{\partial\lambda}[(2D_1+2G\xi^{\cdot})+(2E+2H\xi^{\cdot})+(F+I\xi^{\cdot})]\right)(a+d_1+h) \\ -\frac{(1-\alpha)}{2\xi^3}\left([\,(2-[1+t\xi][t\xi+2]e^{-t\xi})\,\frac{(1-\alpha)}{2\xi^3}(2E+2H\xi^{\cdot})+\frac{2(1-\alpha)\xi^{\cdot}}{\xi^3}[1-(1+\xi t)e^{-t\xi}](F+I\xi^{\cdot})]\right) \end{bmatrix} \quad (228)$$

As $t \to \infty$

$$a = -\frac{1}{\xi^2}. \quad (229)$$

$$b = \frac{-\xi^{\cdot}}{\xi^2}. \quad (230)$$

$$d_1 = \frac{2\lambda}{\xi^3}. \quad (231)$$

$$g = \frac{[\lambda^{\cdot}\xi - 2\lambda\xi^{\cdot}]}{\xi^3}. \quad (232)$$

$$l = \frac{-\xi^{\cdot}}{\xi^2} = b. \quad (233)$$

$$h = \frac{(2\lambda\xi^{\cdot}-\lambda^{\cdot}\xi)}{\xi^3}. \quad (234)$$

$$r = \frac{(\lambda^{\cdot\cdot}\xi^2-\lambda\xi\xi^{\cdot\cdot}-2\lambda^{\cdot}\xi\xi^{\cdot}+2\lambda\xi^{\cdot 2})}{\xi^3}. \quad (235)$$

$$\Delta_{\infty} = (-a(lg-ar)+b(ah-ld_1))) = \frac{\lambda^{\cdot\cdot}\xi^2-\lambda\xi\xi^{\cdot\cdot}-4\lambda^{\cdot}\xi\xi^{\cdot}+4\lambda\xi^{\cdot 2}}{\xi^7}. \quad (236)$$

$$R_{1212,t\to\infty}^{(\alpha)} = \begin{bmatrix} \left(-\frac{(1-\alpha)}{2\xi^3}\frac{\partial}{\partial\lambda}[(2D_1+2G\xi^{\cdot})+(2E+2H\xi^{\cdot})+(F+I\xi^{\cdot})]\right)(a+d_1+h) \\ -\frac{(1-\alpha)}{2\xi^3}\left([\,\frac{(1-\alpha)}{\xi^3}(2E+2H\xi^{\cdot})+\frac{2(1-\alpha)\xi^{\cdot}}{\xi^3}(F+I\xi^{\cdot})]\right) \end{bmatrix}. \quad (237)$$

We have

$$(a+d_1+h) = \left(\frac{2\lambda}{\xi^3}-\frac{1}{\xi^2}+\frac{(2\lambda\xi^{\cdot}-\lambda^{\cdot}\xi)}{\xi^3}\right) = \frac{(2\lambda\xi^{\cdot}-\lambda^{\cdot}\xi+2\lambda-\xi)}{\xi^3}. \quad (236)$$

Recall that:

$$I = (2D_1 + 2G\dot{\xi}) = \frac{(lg-ar)+(ah-ld_1)\dot{\xi}}{\Delta_\infty} = \frac{\xi^2(\ddot{\lambda}\xi^2-\lambda\xi\ddot{\xi}-2\dot{\lambda}\xi\dot{\xi}+4\lambda\dot{\xi}^2)}{\ddot{\lambda}\xi^2-\lambda\xi\ddot{\xi}-4\dot{\lambda}\xi\dot{\xi}+4\lambda\dot{\xi}^2}. \tag{237}$$

$$(2E + 2H\dot{\xi}) = \frac{2l[a\dot{\xi} - b]}{\Delta_\infty} = \frac{-2\dot{\xi}}{\xi^2}\,\frac{[-\frac{1}{\xi^2}\dot{\xi} + \frac{\dot{\xi}}{\xi^2}]}{\Delta_\infty} = 0. \tag{238}$$

$$(F + I\dot{\xi}) = \frac{a(b - a\dot{\xi})}{\Delta_\infty} = \frac{-\frac{1}{\xi^2}(-\frac{1}{\xi^2}\dot{\xi} + \frac{1}{\xi^2}\dot{\xi})}{\Delta_\infty} = 0. \tag{239}$$

Combining *Eq. (236)*, *Eqs. (34)-(36)*, we have

$$\begin{aligned} R^{(\alpha)}_{1212,t\to\infty} &= \left[\left(-\frac{(1-\alpha)(2\lambda\dot{\xi}-\dot{\lambda}\xi+2\lambda-\xi)}{2\xi^4}\frac{\partial}{\partial\lambda}\left[\frac{(\ddot{\lambda}\xi^2-\lambda\xi\ddot{\xi}-2\dot{\lambda}\xi\dot{\xi}+4\lambda\dot{\xi}^2)}{\ddot{\lambda}\xi^2-\lambda\xi\ddot{\xi}-4\dot{\lambda}\xi\dot{\xi}+4\lambda\dot{\xi}^2}\right]\right)\right] \\ &= \left[\left(-\frac{(1-\alpha)(2\lambda\dot{\xi}-\dot{\lambda}\xi+2\lambda-\xi)(4\dot{\xi}^2-\xi\ddot{\xi})}{2\xi^4}\left[\frac{(\ddot{\lambda}\xi^2-\lambda\xi\ddot{\xi}-4\dot{\lambda}\xi\dot{\xi}+4\lambda\dot{\xi}^2)-(\ddot{\lambda}\xi^2-\lambda\xi\ddot{\xi}-2\dot{\lambda}\xi\dot{\xi}+4\lambda\dot{\xi}^2)}{(\ddot{\lambda}\xi^2-\lambda\xi\ddot{\xi}-4\dot{\lambda}\xi\dot{\xi}+4\lambda\dot{\xi}^2)^2}\right]\right)\right] \\ &= \left[\left(\frac{(1-\alpha)(2\lambda\dot{\xi}-\dot{\lambda}\xi+2\lambda-\xi)\dot{\lambda}\dot{\xi}(4\dot{\xi}^2-\xi\ddot{\xi})}{\xi^3(\ddot{\lambda}\xi^2-\lambda\xi\ddot{\xi}-4\dot{\lambda}\xi\dot{\xi}+4\lambda\dot{\xi}^2)^2}\right)\right]. \end{aligned} \tag{240}$$

Based on our calculations, the $\alpha -$ Gaussian Curvature, $K^{(\alpha)}_\infty$ as time approaches infinity is devised by:

$$K^{(\alpha)}_\infty = \frac{R^{(\alpha)}_{1212,t\to\infty}}{\Delta_\infty} = \frac{\left[\left(\frac{(1-\alpha)(2\lambda\dot{\xi}-\dot{\lambda}\xi+2\lambda-\xi)\dot{\lambda}\dot{\xi}(4\dot{\xi}^2-\xi\ddot{\xi})}{\xi^3(\ddot{\lambda}\xi^2-\lambda\xi\ddot{\xi}-4\dot{\lambda}\xi\dot{\xi}+4\lambda\dot{\xi}^2)^2}\right)\right]}{\frac{\ddot{\lambda}\xi^2-\lambda\xi\ddot{\xi}-4\dot{\lambda}\xi\dot{\xi}+4\lambda\dot{\xi}^2}{\xi^7}} =$$

$$\left[\left(\frac{(1-\alpha)\dot{\lambda}\dot{\xi}\xi^4(2\lambda\dot{\xi}-\dot{\lambda}\xi+2\lambda-\xi)(4\dot{\xi}^2-\xi\ddot{\xi})}{(\ddot{\lambda}\xi^2-\lambda\xi\ddot{\xi}-4\dot{\lambda}\xi\dot{\xi}+4\lambda\dot{\xi}^2)^3}\right)\right]$$.(c.f., *Eq. (225)*)

In the following theorem, the zeros of the $\alpha -$ Gaussian Curvature $K^{(\alpha)}_\infty$ are determined. Based on this, the paths of motion of the coordinates at which the underlying QM is looked at as a developable surface are obtained. The following theorem presents a novel approach which unifies IG with Riemannian Geometry, the theory of developable surfaces and the theory of time -dependent queueing systems.

Theorem 19. The underlying M/M/∞ QM is developable on the following trajectories:

$\alpha = 1$, or $\lambda = constant$ or $\xi =$ constant or $\lambda = \xi = 0$ or $\xi = \vartheta_2 e^{4\vartheta_1 t}$ or $\lambda = a_1, (\xi - 2a_1) = a_2 e^{\frac{t}{2a_1}}$ or $\xi = a_3, (2\lambda - a_3) = a_4 e^{-\frac{2t}{a_3}}$.

for any arbitrary non-zero real constants $\vartheta_1, \vartheta_2, a_1, a_2$ and a_3, a_4. **(241)**

Proof: $K^{(\alpha)}_\infty = 0$ if and only if

$$(1-\alpha)\dot{\lambda}\dot{\xi}\xi^4(4\dot{\xi}^2 - \xi\ddot{\xi})(2\lambda\dot{\xi} - \dot{\lambda}\xi + 2\lambda - \xi) = 0. \tag{242}$$

If and only if one of the following equations holds:

$$(1-\alpha) = 0 \Longrightarrow \alpha = 1. \tag{243}$$

$$\dot{\lambda} = 0 \Longrightarrow \lambda = \text{constant}. \tag{244}$$

$$\dot{\xi} = 0 \Longrightarrow \xi = \text{constant}. \tag{245}$$

$$\xi^4 = 0 \Longrightarrow \xi = 0. \tag{246}$$

$$(4\dot{\xi}^2 - \xi\ddot{\xi}) = 0 \Longrightarrow$$

$$4\frac{\dot{\xi}}{\xi} = \frac{\ddot{\xi}}{\dot{\xi}}. \tag{247}$$

Eq. (247) which has a closed form solution

$$4\vartheta_1\xi = \xi^{\cdot}, \tag{248}$$

where ϑ_1 is an arbitrary constant.

Eq. (249) has the closed form solution:

$$\xi = \vartheta_2 e^{4\vartheta_1 t}, \tag{249}$$

where ϑ_2 is an arbitrary constant.

$$(2\lambda\xi^{\cdot} - \lambda^{\cdot}\xi + 2\lambda - \xi) = 0 \tag{250}$$

It is clear that $\lambda = 0$ and $\xi = 0$ are two closed form solutions of *Eq. (250)*. We can try another closed form solutions of *Eq. (251)*, for example, let $\lambda = a_1$ for any arbitrary non-zero real constant a_1. This implies:

$(2a_1\xi^{\cdot} + 2a_1 - \xi) = 0 \Rightarrow 2a_1\xi^{\cdot} = \xi - 2a_1 \Rightarrow \frac{d\xi}{\xi - 2a_1} = \frac{dt}{2a_1} \Rightarrow \ln(\xi - 2a_1) = \frac{t}{2a_1} + \ln a_2$ for some arbitrary non-zero real constant a_2. Therefore, we have the closed form solutions of *Eq. (251)* to be given by

$$\lambda = a_1, (\xi - 2a_1) = a_2 e^{\frac{t}{2a_1}}. \tag{251}$$

Moreover, we can deduce more closed form solutions of *Eq. (251)*, for example, let $\xi = a_3$ for any arbitrary non-zero real constant a_3. This implies:

$(-\lambda^{\cdot}a_3 + 2\lambda - a_3) = 0 \Rightarrow \lambda^{\cdot}a_3 = -(2\lambda - a_3) \Rightarrow \frac{d\lambda}{(2\lambda - a_3)} = -\frac{2dt}{a_3} \Rightarrow ln(2\lambda - a_3) = -\frac{2t}{a_3} + ln a_4$ for some arbitrary non-zero real constant a_4. Therefore, we have the closed form solutions of *Eq. (251)* to be given by

$$\xi = a_3, (2\lambda - a_3) = a_4 e^{-\frac{2t}{a_3}}. \tag{252}$$

12 | RCT, $\underset{\infty}{(\overrightarrow{RCT})}$ and the $\alpha-$Sectional Gaussian Curvatures of M/M/∞ QM F M/M/∞ QM As Time Approaches Infinity

In this section, the RCT of M/M/∞ is obtained. These calculations are needed in the following sections.

$$\underset{\infty}{(\overrightarrow{RCT})} = \begin{bmatrix} R^{(\alpha)}_{11,\infty} & R^{(\alpha)}_{12,\infty} & R^{(\alpha)}_{13,\infty} \\ R^{(\alpha)}_{21,\infty} & R^{(\alpha)}_{22,\infty} & R^{(\alpha)}_{23,\infty} \\ R^{(\alpha)}_{31,\infty} & R^{(\alpha)}_{32,\infty} & R^{(\alpha)}_{33,\infty} \end{bmatrix}.$$

Which is a nine-dimensional tensor.

12.1 | The First Component, $R^{(\alpha)}_{11,\infty}$

Theorem 20. For the underlying M/M/∞ QM, the first component of the RCT as time reaches infinity, $R^{(\alpha)}_{11,\infty}$.

$$\begin{aligned} R^{(\alpha)}_{11,\infty} = \frac{(1-\alpha)(4\xi^{\cdot 2} - \xi\xi^{\cdot\cdot})\xi^{\cdot}\lambda^{\cdot}}{(\lambda^{\cdot\cdot}\xi^2 - \lambda\xi\xi^{\cdot\cdot} - 4\lambda^{\cdot}\xi\xi^{\cdot} + 4\lambda\xi^{\cdot 2})^3}&([-\xi^{\cdot 2}(2\lambda\xi^{\cdot} - \lambda^{\cdot}\xi + 2\lambda - \xi)] - [2\lambda^{\cdot}\xi(1+\xi^{\cdot})] \\ &+ [2(\xi^{\cdot}+1)(\lambda^{\cdot\cdot}\xi^2 - \lambda\xi\xi^{\cdot\cdot} - 3\lambda^{\cdot}\xi\xi^{\cdot} + 2\lambda\xi^{\cdot 2})] - [\xi^{\cdot 2}\xi^4] \\ &+ [2\xi^{\cdot}(-2\lambda\xi^{\cdot} + \lambda^{\cdot}\xi - 2\lambda + \xi)] + [\xi^{\cdot}\xi^3]). \end{aligned}$$

Proof: recalling from *Eq. (228)-(234)*, that

$-\frac{1}{\xi^2}, b = \frac{-\xi^{\cdot}}{\xi^2}, d_1 = \frac{2\lambda}{\xi^3}, g = \frac{[\lambda^{\cdot}\xi - 2\lambda\xi^{\cdot}]}{\xi^3}, l = \frac{-\xi^{\cdot}}{\xi^2} = b, h = \frac{(2\lambda\xi^{\cdot} - \lambda^{\cdot}\xi)}{\xi^3}, r = \frac{(\lambda^{\cdot\cdot}\xi^2 - \lambda\xi\xi^{\cdot\cdot} - 2\lambda^{\cdot}\xi\xi^{\cdot} + 2\lambda\xi^{\cdot 2})}{\xi^3},$

$= (-a(lg - ar) + b(ah - ld_1))) = \frac{\lambda^{\cdot\cdot}\xi^2 - \lambda\xi\xi^{\cdot\cdot} - 4\lambda^{\cdot}\xi\xi^{\cdot} + 4\lambda\xi^{\cdot 2}}{\xi^7}.$

We have

$$(a + d_1 + h) = (\frac{2\lambda}{\xi^3} - \frac{1}{\xi^2} + \frac{(2\lambda\xi^{\cdot} - \lambda^{\cdot}\xi)}{\xi^3}) = \frac{(2\lambda\xi^{\cdot} - \lambda^{\cdot}\xi + 2\lambda - \xi)}{\xi^3}. \quad (253)$$

$$R^{(\alpha)}_{11,\infty} = R^{(\alpha)}_{1111,\infty}g^{11,\infty} + R^{(\alpha)}_{1211,\infty}g^{21,\infty} + R^{(\alpha)}_{1311,\infty}g^{31,\infty} + R^{(\alpha)}_{1112,\infty}g^{12,\infty} + R^{(\alpha)}_{1113}g^{13,\infty} + R^{(\alpha)}_{1212,\infty}g^{22,\infty} + R^{(\alpha)}_{1213,\infty}g^{23,\infty} + R^{(\alpha)}_{1312,\infty}g^{32,\infty} + R^{(\alpha)}_{1313,\infty}g^{33,\infty}. \quad (254)$$

We have

$$R^{(\alpha)}_{ijkl} = \left[\left(\partial_j\Gamma^{s(\alpha)}_{ik} - \partial_i\Gamma^{s(\alpha)}_{jk}\right)g_{sl} + \left(\Gamma^{(\alpha)}_{j\beta,l}\Gamma^{\beta(\alpha)}_{ik} - \Gamma^{(\alpha)}_{i\beta,l}\Gamma^{\beta(\alpha)}_{jk}\right)\right].$$

where $\Gamma^{k(\alpha)}_{ij} = \Gamma^{(\alpha)}_{ij,s}g^{sk}, i, j, k, s = 1,2,\ldots,n.$

$$R^{(\alpha)}_{1111,t\to\infty} = \left[\left(\partial_1\Gamma^{s(\alpha)}_{11} - \partial_1\Gamma^{s(\alpha)}_{11}\right)g_{s1} + \left(\Gamma^{(\alpha)}_{1\beta,1}\Gamma^{\beta(\alpha)}_{11} - \Gamma^{(\alpha)}_{1\beta,1}\Gamma^{\beta(\alpha)}_{11}\right)\right] = 0. \quad (255)$$

$$R^{(\alpha)}_{1113,t\to\infty} = \left[\left(\partial_1\Gamma^{s(\alpha)}_{11} - \partial_1\Gamma^{s(\alpha)}_{11}\right)g_{s3} + \left(\Gamma^{(\alpha)}_{1\beta,3}\Gamma^{\beta(\alpha)}_{11} - \Gamma^{(\alpha)}_{1\beta,3}\Gamma^{\beta(\alpha)}_{11}\right)\right] = 0 = R^{(\alpha)}_{1112,\infty}. \quad (256)$$

$$\begin{aligned}R^{(\alpha)}_{1211,t\to\infty} &= \left[\left(\partial_2\Gamma^{s(\alpha)}_{11} - \partial_1\Gamma^{s(\alpha)}_{21}\right)g_{s1} + \left(\Gamma^{(\alpha)}_{2\beta,l}\Gamma^{\beta(\alpha)}_{11} - \Gamma^{(\alpha)}_{1\beta,l}\Gamma^{\beta(\alpha)}_{21}\right)\right] \\ &= \left[\left(\frac{\partial}{\partial\xi}(\Gamma^{1(\alpha)}_{11} + \Gamma^{2(\alpha)}_{11} + \Gamma^{3(\alpha)}_{11}) - \frac{\partial}{\partial\lambda}(\Gamma^{1(\alpha)}_{21} + \Gamma^{2(\alpha)}_{21} + \Gamma^{3(\alpha)}_{21})\right) + \left(\Gamma^{(\alpha)}_{2\beta,1}\Gamma^{\beta(\alpha)}_{11} - \Gamma^{(\alpha)}_{1\beta,1}\Gamma^{\beta(\alpha)}_{21}\right)\right]\end{aligned} \quad (257)$$

$$= \left[\left(\frac{\partial}{\partial\xi}(0) - \frac{\partial}{\partial\lambda}(\frac{(1-\alpha)}{2\xi^3}(2D_1 + 2G\xi^{\cdot}) + \frac{(1-\alpha)}{2\xi^3}(2E + 2H\xi^{\cdot}) + \frac{(1-\alpha)}{2\xi^3}(2F + 2I\xi^{\cdot})\right) - (0)\right] = \frac{(\alpha-1)}{\xi^3}\left[\left(\frac{\partial}{\partial\lambda}(D_1 + E + F + (H + G + I)\xi^{\cdot})\right)\right].$$

$$\begin{aligned}\left[\left(\frac{\partial}{\partial\lambda}(D_1 + E + F + (H + G + I)\xi^{\cdot})\right)\right] &= \xi^2\frac{\partial}{\partial\lambda}\left[\frac{(\lambda^{\cdot\cdot}\xi^2 - \lambda\xi\xi^{\cdot\cdot} - 2\lambda^{\cdot}\xi\xi^{\cdot} + 4\lambda\xi^{\cdot 2})}{\lambda^{\cdot\cdot}\xi^2 - \lambda\xi\xi^{\cdot\cdot} - 4\lambda^{\cdot}\xi\xi^{\cdot} + 4\lambda\xi^{\cdot 2}}\right] \\ &= \left[\frac{2\lambda^{\cdot}\xi^{\cdot}(\xi\xi^{\cdot\cdot} - 4\xi^{\cdot 2})\xi^3}{(\lambda^{\cdot\cdot}\xi^2 - \lambda\xi\xi^{\cdot\cdot} - 4\lambda^{\cdot}\xi\xi^{\cdot} + 4\lambda\xi^{\cdot 2})^2}\right].\end{aligned} \quad (258)$$

Hence,

$$R^{(\alpha)}_{1211,t\to\infty} = \left[\frac{2(1-\alpha)\lambda^{\cdot}\xi^{\cdot}(\xi\xi^{\cdot\cdot} - 4\xi^{\cdot 2})(\xi^{\cdot} + 1)}{\xi^2(\lambda^{\cdot\cdot}\xi^2 - \lambda\xi\xi^{\cdot\cdot} - 4\lambda^{\cdot}\xi\xi^{\cdot} + 4\lambda\xi^{\cdot 2})^2}\right]. \quad (259)$$

$$\begin{aligned}R^{(\alpha)}_{1311,t\to\infty} &= \left[\left(\partial_3\Gamma^{s(\alpha)}_{11} - \partial_1\Gamma^{s(\alpha)}_{21}\right)g_{s1} + \left(\Gamma^{(\alpha)}_{3\beta,1}\Gamma^{\beta(\alpha)}_{11} - \Gamma^{(\alpha)}_{1\beta,1}\Gamma^{\beta(\alpha)}_{21}\right)\right] \\ &= \left[\frac{2(1-\alpha)\lambda^{\cdot}\xi^{\cdot}(\xi\xi^{\cdot\cdot} - 4\xi^{\cdot 2})(1 + \xi^{\cdot})}{\xi^2(\lambda^{\cdot\cdot}\xi^2 - \lambda\xi\xi^{\cdot\cdot} - 4\lambda^{\cdot}\xi\xi^{\cdot} + 4\lambda\xi^{\cdot 2})^2}\right].\end{aligned} \quad (260)$$

$$R^{(\alpha)}_{1312,t\to\infty} = \frac{2(1-\alpha)}{\xi^3}\left[\frac{\lambda^{\cdot}\xi^{\cdot}(\xi\xi^{\cdot\cdot} - 4\xi^{\cdot 2})(2\lambda\xi^{\cdot} - \lambda^{\cdot}\xi + 2\lambda - \xi)}{(\lambda^{\cdot\cdot}\xi^2 - \lambda\xi\xi^{\cdot\cdot} - 4\lambda^{\cdot}\xi\xi^{\cdot} + 4\lambda\xi^{\cdot 2})^2}\right]. \quad (261)$$

$$R^{(\alpha)}_{1213,t\to\infty} = \frac{(1-\alpha)}{\xi^2}\left[\frac{\lambda^{\cdot}\xi^{\cdot 2}(\xi\xi^{\cdot\cdot} - 4\xi^{\cdot 2})\xi^3}{(\lambda^{\cdot\cdot}\xi^2 - \lambda\xi\xi^{\cdot\cdot} - 4\lambda^{\cdot}\xi\xi^{\cdot} + 4\lambda\xi^{\cdot 2})^2}\right]. \quad (262)$$

$$R^{(\alpha)}_{1313,t\to\infty} = \frac{(1-\alpha)}{\xi^2}\left[\frac{\lambda^{\cdot}\xi^{\cdot 2}(\xi\xi^{\cdot\cdot} - 4\xi^{\cdot 2})\xi^3}{(\lambda^{\cdot\cdot}\xi^2 - \lambda\xi\xi^{\cdot\cdot} - 4\lambda^{\cdot}\xi\xi^{\cdot} + 4\lambda\xi^{\cdot 2})^2}\right]. \quad (263)$$

Therefore,

$$R_{11,\infty}^{(\alpha)} = R_{1211,\infty}^{(\alpha)} D_1 + R_{1311,\infty}^{(\alpha)} G + R_{1212,\infty}^{(\alpha)} E + R_{1213,\infty}^{(\alpha)} F + R_{1312,\infty}^{(\alpha)} H + R_{1313,\infty}^{(\alpha)} I.$$

$$\begin{aligned} R_{11,\infty}^{(\alpha)} = (&\left[\left(\frac{(1-\alpha)(2\lambda\dot{\xi} - \lambda\dot{\xi} + 2\lambda - \xi)\lambda\dot{\xi}(4\dot{\xi}^2 - \xi\ddot{\xi})}{\xi^3(\lambda\ddot{\xi}^2 - \lambda\xi\ddot{\xi} - 4\lambda\dot{\xi}\dot{\xi} + 4\lambda\dot{\xi}^2)^2}\right)\right] E \\ &+ \left[\frac{2(1-\alpha)\lambda\dot{\xi}(\xi\ddot{\xi} - 4\dot{\xi}^2)(1+\dot{\xi})}{\xi^2(\lambda\ddot{\xi}^2 - \lambda\xi\ddot{\xi} - 4\lambda\dot{\xi}\dot{\xi} + 4\lambda\dot{\xi}^2)^2}\right] G \\ &+ \left[\frac{2(1-\alpha)\lambda\dot{\xi}(\xi\ddot{\xi} - 4\dot{\xi}^2)(\dot{\xi}+1)}{\xi^2(\lambda\ddot{\xi}^2 - \lambda\xi\ddot{\xi} - 4\lambda\dot{\xi}\dot{\xi} + 4\lambda\dot{\xi}^2)^2}\right] D_1 \\ &+ \frac{(1-\alpha)}{\xi^2}\left[\frac{\lambda\dot{\xi}^2(\xi\ddot{\xi} - 4\dot{\xi}^2)\xi^3}{(\lambda\ddot{\xi}^2 - \lambda\xi\ddot{\xi} - 4\lambda\dot{\xi}\dot{\xi} + 4\lambda\dot{\xi}^2)^2}\right] F \\ &+ \frac{2(1-\alpha)}{\xi^3}\left[\frac{\lambda\dot{\xi}(\xi\ddot{\xi} - 4\dot{\xi}^2)(2\lambda\dot{\xi} - \lambda\dot{\xi} + 2\lambda - \xi)}{(\lambda\ddot{\xi}^2 - \lambda\xi\ddot{\xi} - 4\lambda\dot{\xi}\dot{\xi} + 4\lambda\dot{\xi}^2)^2}\right] H \\ &+ \frac{(1-\alpha)}{\xi^2}\left[\frac{\lambda\dot{\xi}^2(\xi\ddot{\xi} - 4\dot{\xi}^2)\xi^3}{(\lambda\ddot{\xi}^2 - \lambda\xi\ddot{\xi} - 4\lambda\dot{\xi}\dot{\xi} + 4\lambda\dot{\xi}^2)^2}\right] I). \end{aligned} \quad (264)$$

$$E = \frac{(-lb)}{\Delta_\infty} = \frac{-\xi^3\dot{\xi}^2}{\lambda\ddot{\xi}^2 - \lambda\xi\ddot{\xi} - 4\lambda\dot{\xi}\dot{\xi} + 4\lambda\dot{\xi}^2}. \quad (265)$$

$$G = \frac{(ah - ld_1)}{\Delta_\infty} = \frac{\lambda\dot{\xi}^3}{\lambda\ddot{\xi}^2 - \lambda\xi\ddot{\xi} - 4\lambda\dot{\xi}\dot{\xi} + 4\lambda\dot{\xi}^2}. \quad (266)$$

$$D_1 = \frac{(lg - ar)}{\Delta_\infty} = \frac{(\lambda\ddot{\xi}^2 - \lambda\xi\ddot{\xi} - 3\lambda\dot{\xi}\dot{\xi} + 2\lambda\dot{\xi}^2)\xi^2}{\lambda\ddot{\xi}^2 - \lambda\xi\ddot{\xi} - 4\lambda\dot{\xi}\dot{\xi} + 4\lambda\dot{\xi}^2}. \quad (267)$$

$$F = \frac{(ab)}{\Delta_\infty} = \frac{\xi^3\dot{\xi}}{\lambda\ddot{\xi}^2 - \lambda\xi\ddot{\xi} - 4\lambda\dot{\xi}\dot{\xi} + 4\lambda\dot{\xi}^2}. \quad (268)$$

$$H = \frac{(la)}{\Delta_\infty} = \frac{\xi^3\dot{\xi}}{\lambda\ddot{\xi}^2 - \lambda\xi\ddot{\xi} - 4\lambda\dot{\xi}\dot{\xi} + 4\lambda\dot{\xi}^2}. \quad (269)$$

$$I = \frac{(-a^2)}{\Delta_\infty} = -\frac{\xi^3}{\lambda\ddot{\xi}^2 - \lambda\xi\ddot{\xi} - 4\lambda\dot{\xi}\dot{\xi} + 4\lambda\dot{\xi}^2}. \quad (270)$$

Based on the above complicated calculations, we have

$$\begin{aligned} R_{11,\infty}^{(\alpha)} = \Bigg(&\left[\left(\frac{(1-\alpha)(2\lambda\dot{\xi} - \lambda\dot{\xi} + 2\lambda - \xi)\lambda\dot{\xi}(4\dot{\xi}^2 - \xi\ddot{\xi})}{(\lambda\ddot{\xi}^2 - \lambda\xi\ddot{\xi} - 4\lambda\dot{\xi}\dot{\xi} + 4\lambda\dot{\xi}^2)^2}\right)\right]\left[\frac{-\dot{\xi}^2}{\lambda\ddot{\xi}^2 - \lambda\xi\ddot{\xi} - 4\lambda\dot{\xi}\dot{\xi} + 4\lambda\dot{\xi}^2}\right] \\ &+ \left[\frac{2(1-\alpha)\lambda\dot{\xi}(\xi\ddot{\xi} - 4\dot{\xi}^2)(1+\dot{\xi})}{(\lambda\ddot{\xi}^2 - \lambda\xi\ddot{\xi} - 4\lambda\dot{\xi}\dot{\xi} + 4\lambda\dot{\xi}^2)^2}\right]\left[\frac{\lambda\dot{\xi}}{\lambda\ddot{\xi}^2 - \lambda\xi\ddot{\xi} - 4\lambda\dot{\xi}\dot{\xi} + 4\lambda\dot{\xi}^2}\right] \\ &+ \left[\frac{2(1-\alpha)\lambda\dot{\xi}(\xi\ddot{\xi} - 4\dot{\xi}^2)(\dot{\xi}+1)}{(\lambda\ddot{\xi}^2 - \lambda\xi\ddot{\xi} - 4\lambda\dot{\xi}\dot{\xi} + 4\lambda\dot{\xi}^2)^2}\right]\left[\frac{(\lambda\ddot{\xi}^2 - \lambda\xi\ddot{\xi} - 3\lambda\dot{\xi}\dot{\xi} + 2\lambda\dot{\xi}^2)}{\lambda\ddot{\xi}^2 - \lambda\xi\ddot{\xi} - 4\lambda\dot{\xi}\dot{\xi} + 4\lambda\dot{\xi}^2}\right] \\ &+ (1-\alpha)\left[\frac{\lambda\dot{\xi}^2(\xi\ddot{\xi} - 4\dot{\xi}^2)\xi^4}{(\lambda\ddot{\xi}^2 - \lambda\xi\ddot{\xi} - 4\lambda\dot{\xi}\dot{\xi} + 4\lambda\dot{\xi}^2)^2}\right]\left[\frac{\dot{\xi}}{\lambda\ddot{\xi}^2 - \lambda\xi\ddot{\xi} - 4\lambda\dot{\xi}\dot{\xi} + 4\lambda\dot{\xi}^2}\right] \\ &+ 2(1-\alpha)\left[\frac{\lambda\dot{\xi}(\xi\ddot{\xi} - 4\dot{\xi}^2)(2\lambda\dot{\xi} - \lambda\dot{\xi} + 2\lambda - \xi)}{(\lambda\ddot{\xi}^2 - \lambda\xi\ddot{\xi} - 4\lambda\dot{\xi}\dot{\xi} + 4\lambda\dot{\xi}^2)^2}\right]\left[\frac{\dot{\xi}}{\lambda\ddot{\xi}^2 - \lambda\xi\ddot{\xi} - 4\lambda\dot{\xi}\dot{\xi} + 4\lambda\dot{\xi}^2}\right] \\ &+ (1-\alpha)\left[\frac{\lambda\dot{\xi}^2(\xi\ddot{\xi} - 4\dot{\xi}^2)\xi^3}{(\lambda\ddot{\xi}^2 - \lambda\xi\ddot{\xi} - 4\lambda\dot{\xi}\dot{\xi} + 4\lambda\dot{\xi}^2)^2}\right]\left[-\frac{1}{\lambda\ddot{\xi}^2 - \lambda\xi\ddot{\xi} - 4\lambda\dot{\xi}\dot{\xi} + 4\lambda\dot{\xi}^2}\right]\Bigg). \end{aligned} \quad (271)$$

$$= \frac{(1-\alpha)}{(\lambda^{\cdot\cdot}\xi^2 - \lambda\xi\xi^{\cdot\cdot} - 4\lambda^{\cdot}\xi\xi^{\cdot} + 4\lambda\xi^{\cdot 2})^3}([-\xi^{\cdot 2}(2\lambda\xi^{\cdot} - \lambda^{\cdot}\xi + 2\lambda - \xi)\lambda^{\cdot}\xi^{\cdot}(4\xi^{\cdot 2} - \xi\xi^{\cdot\cdot})]$$
$$+ [2\lambda^{\cdot 2}\xi\xi^{\cdot}(\xi\xi^{\cdot\cdot} - 4\xi^{\cdot 2})(1+\xi^{\cdot})]$$
$$+ [2\lambda^{\cdot}\xi^{\cdot}(\xi\xi^{\cdot\cdot} - 4\xi^{\cdot 2})(\xi^{\cdot}+1)(\lambda^{\cdot\cdot}\xi^2 - \lambda\xi\xi^{\cdot\cdot} - 3\lambda^{\cdot}\xi\xi^{\cdot} + 2\lambda\xi^{\cdot 2})]$$
$$+ [\lambda^{\cdot}\xi^{\cdot 3}(\xi\xi^{\cdot\cdot} - 4\xi^{\cdot 2})\xi^4] + [2\lambda^{\cdot}\xi^{\cdot 2}(\xi\xi^{\cdot\cdot} - 4\xi^{\cdot 2})(2\lambda\xi^{\cdot} - \lambda^{\cdot}\xi + 2\lambda - \xi)]$$
$$- [\lambda^{\cdot}\xi^{\cdot 2}(\xi\xi^{\cdot\cdot} - 4\xi^{\cdot 2})\xi^3]).$$
$$= \frac{(1-\alpha)(4\xi^{\cdot 2} - \xi\xi^{\cdot\cdot})\xi^{\cdot}\lambda^{\cdot}}{(\lambda^{\cdot\cdot}\xi^2 - \lambda\xi\xi^{\cdot\cdot} - 4\lambda^{\cdot}\xi\xi^{\cdot} + 4\lambda\xi^{\cdot 2})^3}([-\xi^{\cdot 2}(2\lambda\xi^{\cdot} - \lambda^{\cdot}\xi + 2\lambda - \xi)] - [2\lambda^{\cdot}\xi(1+\xi^{\cdot})]$$
$$+ [2(\xi^{\cdot}+1)(\lambda^{\cdot\cdot}\xi^2 - \lambda\xi\xi^{\cdot\cdot} - 3\lambda^{\cdot}\xi\xi^{\cdot} + 2\lambda\xi^{\cdot 2})] - [\xi^{\cdot 2}\xi^4]$$
$$+ [2\xi^{\cdot}(-2\lambda\xi^{\cdot} + \lambda^{\cdot}\xi - 2\lambda + \xi)] + [\xi^{\cdot}\xi^3]).$$

This completes the proof of our theorem.

Remember that the Riemannian manifold is more positively curved than a sphere and has a smaller diameter whenever the RC is positive, according to the Bonnet Myers theorem [10]. Using this information, we will launch a new way of thinking that creates a decision-making process for the situation in which the manifold's diameter is less and the underlying QM is more positively curved than a sphere.

This could be obtained by calculating the zeros of the first component of the RCT as time approaches infinity, $(\overrightarrow{RCT})_{\infty}$, namely, $R^{(\alpha)}_{11,\infty}$. Stated otherwise, this open question is resolved by the subsequent theorem.

Theorem 21. The zeros of $R^{(\alpha)}_{11,\infty}$ (*Eq. (274)*) is one of the following:

I. α=1, in other words, whenever the curvature parameter is set to unity.

II. The equation of motion of the Poissonian arrival rate, ξ is governed by the temporal path.

$\xi = \vartheta_2 e^{4\vartheta_1 t}$. *Eq. (249)*

III. ξ=constant.

IV. λ=constant .

V. The dynamics of both of arrival rate and the Poissonian arrival rate is governed by the equation.

$$\lambda = c_1\xi e^{\frac{t}{\xi}} + c_2,\ \xi = \text{constant}, c_1 \text{ and } c_2 \text{ are any two non-zero real arbitrary constants.} \tag{272}$$

VI. The dynamics of both of arrival rate and the Poissonian arrival rate is governed by the equation.

$$\xi = \tan\left[\frac{\left(t - \frac{\xi^2}{2} + c_3\right)\sqrt{2}}{5}\right], \lambda = \text{zero and } c_3 \text{ is any non-zero real arbirary constant.} \tag{273}$$

Proof: we have

$$R^{(\alpha)}_{11,\infty} = \frac{(1-\alpha)(4\xi^{\cdot 2} - \xi\xi^{\cdot\cdot})\xi^{\cdot}\lambda^{\cdot}}{(\lambda^{\cdot\cdot}\xi^2 - \lambda\xi\xi^{\cdot\cdot} - 4\lambda^{\cdot}\xi\xi^{\cdot} + 4\lambda\xi^{\cdot 2})^3}([-\xi^{\cdot 2}(2\lambda\xi^{\cdot} - \lambda^{\cdot}\xi + 2\lambda - \xi)] - [2\lambda^{\cdot}\xi(1+\xi^{\cdot})] +$$
$$[2(\xi^{\cdot}+1)(\lambda^{\cdot\cdot}\xi^2 - \lambda\xi\xi^{\cdot\cdot} - 3\lambda^{\cdot}\xi\xi^{\cdot} + 2\lambda\xi^{\cdot 2})] - [\xi^{\cdot 2}\xi^4] + [2\xi^{\cdot}(-2\lambda\xi^{\cdot} + \lambda^{\cdot}\xi - 2\lambda + \xi)] + [\xi^{\cdot}\xi^3]).$$

Clearly, $R^{(\alpha)}_{11,\infty} = 0$ if and only if one of the following statements holds:

$$(1-\alpha) = 0. \tag{274}$$
$$(4\xi^{\cdot 2} - \xi\xi^{\cdot\cdot}) = 0. \tag{275}$$
$$\xi^{\cdot} = 0. \tag{276}$$
$$\lambda^{\cdot} = 0. \tag{277}$$
$$([-\xi^{\cdot 2}(2\lambda\xi^{\cdot} - \lambda^{\cdot}\xi + 2\lambda - \xi)] - [2\lambda^{\cdot}\xi(1+\xi^{\cdot})] + [2(\xi^{\cdot}+1)(\lambda^{\cdot\cdot}\xi^2 - \lambda\xi\xi^{\cdot\cdot} - 3\lambda^{\cdot}\xi\xi^{\cdot} + 2\lambda\xi^{\cdot 2})] - [\xi^{\cdot 2}\xi^4] + [2\xi^{\cdot}(-2\lambda\xi^{\cdot} + \lambda^{\cdot}\xi - 2\lambda + \xi)] + [\xi^{\cdot}\xi^3]) = 0. \tag{278}$$

Clearly *Eq. (248)* implies I.

It has been proven that the exact solution of differential *Eq. (248)* is determined by

$\xi = \vartheta_2 e^{4\vartheta_1 t}$. (*Eq. (250)*)

This proves II.

The proofs of III and IV are straightforward.

Following

$$([-\xi^{\cdot 2}(2\lambda\xi^{\cdot} - \lambda^{\cdot}\xi + 2\lambda - \xi)] - [2\lambda^{\cdot}\xi(1+\xi^{\cdot})] + [2(\xi^{\cdot}+1)(\lambda^{\cdot\cdot}\xi^2 - \lambda\xi\xi^{\cdot\cdot} - 3\lambda^{\cdot}\xi\xi^{\cdot} + 2\lambda\xi^{\cdot 2})] - [\xi^{\cdot 2}\xi^4] + [2\xi^{\cdot}(-2\lambda\xi^{\cdot} + \lambda^{\cdot}\xi - 2\lambda + \xi)] + [\xi^{\cdot}\xi^3]) = 0. \quad (279)$$ (c.f., *Eq. (281)*)

Let $\xi = \text{constant}$. This reduces *Eq. (251)* to

$$2\xi(\lambda^{\cdot\cdot}\xi - \lambda^{\cdot}) = 0. \quad (280)$$

which implies $\xi = 0$ or

$$\lambda^{\cdot\cdot}\xi = \lambda^{\cdot}. \quad (281)$$

(equivalently, $\frac{d\lambda^{\cdot}}{\lambda^{\cdot}} = \frac{dt}{\xi}$). These yields $\lambda^{\cdot} = c_1 e^{\frac{t}{\xi}}$, which has a closed form solution:

$\lambda = c_1 \xi e^{\frac{t}{\xi}} + c_2$. (*Eq. (275)*)

with non − zero real constants c_1, c_2

This proves 5).

Moreover, plugging $\lambda = 0$ in *Eq. (252)* yields

$$0 = [(-5\xi\xi^{\cdot 2}) - \xi^{\cdot 2}\xi^4 + 2\xi\xi^{\cdot} + \xi^{\cdot}\xi^3] = \xi\xi^{\cdot}[(-5\xi\xi^{\cdot}) - \xi^{\cdot}\xi^4 + 2\xi + \xi^3]. \quad (282)$$

It can be easily seen that (282) generates $\xi = 0$ or $\xi = \text{constant}$

or

$[(-5\xi^{\cdot}) - \xi^{\cdot}\xi^3 + 2 + \xi^2] = 0$, implying

$$\xi^{\cdot} = \frac{2+\xi^2}{5+\xi^3} \Rightarrow \left(\frac{\frac{5}{2}d\xi}{1+(\frac{\xi}{\sqrt{2}})^2} - 2\frac{d\xi}{2+\xi^2}\right) + \xi d\xi = dt. \quad (283)$$

Integrating both sides of *Eq. (283)*, we have the exact solution of of the form

$$\frac{5}{\sqrt{2}}\tan^{-1}\left(\frac{\xi}{\sqrt{2}}\right) + \frac{\xi^2}{2} = (t + c_3) \text{ or } \frac{5}{\sqrt{2}}\tan^{-1}\left(\frac{\xi}{\sqrt{2}}\right) = \left(t - \frac{\xi^2}{2} + c_3\right), c_3 \text{ is any non} - \text{zero real constant.} \quad (284)$$

This rewrites *Eq. (284)* to the compact form

$$\xi = tan\left[\frac{\left(t - \frac{\xi^2}{2} + c_3\right)\sqrt{2}}{5}\right].$$ (c.f., *Eq. (276)*)

This proves V.

11.3 | The α − Sectional curvatures , $K^{(\alpha)}_{ijij} = \frac{R^{(\alpha)}_{ijij}}{(g_{ii})(g_{jj})-(g_{ij})^2}$, $i, j = 1, 2, \ldots, n$

The α -Sectional Curvatures as time approaches infinity are given by

$$K^{(\alpha)}_{ijij,\infty} = \frac{R^{(\alpha)}_{ijij,\infty}}{\Delta_\infty}. \tag{285}$$

The reader can easily check that

$$K^{(\alpha)}_{1111,\infty} = 0 = K^{(\alpha)}_{1112,\infty} = K^{(\alpha)}_{1113,\infty}. \tag{286}$$

$$\Delta_\infty = (-a(lg - ar) + b(ah - ld_1))) = \frac{\ddot{\lambda}\xi^2 - \lambda\xi\ddot{\xi} - 4\dot{\lambda}\xi\dot{\xi} + 4\lambda\dot{\xi}^2}{\xi^7}$$

$$K^{(\alpha)}_{1211,\infty} = \frac{R^{(\alpha)}_{1211,\infty}}{\Delta_\infty} = \left[\frac{2(1-\alpha)\dot{\lambda}\dot{\xi}(\xi\ddot{\xi} - 4\dot{\xi}^2)(\dot{\xi} + 1)\xi^5}{(\ddot{\lambda}\xi^2 - \lambda\xi\ddot{\xi} - 4\dot{\lambda}\xi\dot{\xi} + 4\lambda\dot{\xi}^2)^3}\right]. \tag{287}$$

$$K^{(\alpha)}_{1311,\infty} = \frac{R^{(\alpha)}_{1311,\infty}}{\Delta_\infty} = \left[\frac{2(1-\alpha)\dot{\lambda}\dot{\xi}(\xi\ddot{\xi} - 4\dot{\xi}^2)(1 + \dot{\xi})\xi^5}{(\ddot{\lambda}\xi^2 - \lambda\xi\ddot{\xi} - 4\dot{\lambda}\xi\dot{\xi} + 4\lambda\dot{\xi}^2)^2}\right]. \tag{288}$$

$$K^{(\alpha)}_{1312,\infty} = \frac{R^{(\alpha)}_{1312,\infty}}{\Delta_\infty} = \left[\frac{2(1-\alpha)\dot{\lambda}\dot{\xi}(\xi\ddot{\xi} - 4\dot{\xi}^2)(2\lambda\dot{\xi} - \dot{\lambda}\xi + 2\lambda - \xi)\xi^4}{(\ddot{\lambda}\xi^2 - \lambda\xi\ddot{\xi} - 4\dot{\lambda}\xi\dot{\xi} + 4\lambda\dot{\xi}^2)^2}\right]. \tag{289}$$

$$K^{(\alpha)}_{1311,\infty} = \frac{R^{(\alpha)}_{1311,\infty}}{\Delta_\infty} = \left[\frac{2(1-\alpha)\dot{\lambda}\dot{\xi}(\xi\ddot{\xi} - 4\dot{\xi}^2)(1 + \dot{\xi})\xi^5}{(\ddot{\lambda}\xi^2 - \lambda\xi\ddot{\xi} - 4\dot{\lambda}\xi\dot{\xi} + 4\lambda\dot{\xi}^2)^2}\right]. \tag{290}$$

$$K^{(\alpha)}_{1213,\infty} = \frac{R^{(\alpha)}_{1213,\infty}}{\Delta_\infty} = \left[\frac{(1-\alpha)\dot{\lambda}\dot{\xi}^2(\xi\ddot{\xi} - 4\dot{\xi}^2)\xi^8}{(\ddot{\lambda}\xi^2 - \lambda\xi\ddot{\xi} - 4\dot{\lambda}\xi\dot{\xi} + 4\lambda\dot{\xi}^2)^2}\right]. \tag{291}$$

$$K^{(\alpha)}_{1313,\infty} = \frac{R^{(\alpha)}_{1313,\infty}}{\Delta_\infty} = \left[\frac{(1-\alpha)\dot{\lambda}\dot{\xi}^2(\xi\ddot{\xi} - 4\dot{\xi}^2)\xi^3}{(\ddot{\lambda}\xi^2 - \lambda\xi\ddot{\xi} - 4\dot{\lambda}\xi\dot{\xi} + 4\lambda\dot{\xi}^2)^2}\right]. \tag{292}$$

The remaining 72 α-Sectional Curvatures can be obtained by following the same procedure.

13 | Conclusion

The research done for this paper offers a fresh method for modelling the IG of a queuing system. From the perspective of IG, the manifold of the temporary M/M/∞ queue is described in this context.

In summary, the current work provides a giant step ahead to the establishment of the contemporary theory of relativistic IG of transient queues. There are several avenues of future work. To start with, we are going to extend this novel approach to many existing transient queueing systems. Also, another interesting path is to investigate the Information geometric analysis of the overall dynamic of time dependent distributions in Physics and Quantum Mechanics.

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Conflicts of Interest

The author declare no conflict of interest.

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