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The Theory of Complex-Valued Data Envelopment Analysis (DEA)

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Abstract

The traditional Data Envelopment Analysis (DEA) models assumes real-valued inputs and outputs. In many occasions, some inputs and outputs can only take non crisp values. Several studies have been made to deal with uncertain input and output data in DEA. The question is related to the occasions of complex-valued data. According to the major application of complex numbers in engineering, this paper introduces a model of DEA in the context of complex-valued data and sketch both radial and non-radial models. Furthermore, the study reveals a ranking index for sorting the units in complex environments. To elucidate the efficacy of the proposed approach a real case of an electric circuit is examined. The results exhibit the usefulness of the suggested approach and point out that the models have practical outcomes for decision-makers.

Keywords: Complex-valued data set, Data envelopment analysis, Performance evaluation, Electric circuits, Ranking index.

1 | Introduction

Data Envelopment Analysis (DEA), originated by Charnes et al. [1], is a non-parametric data driven mathematical method that has been developed for evaluating the efficiency of a set of similar Decision-Making Units (DMUs) with multiple inputs and multiple outputs. The Charnes, Cooper, and Rhodes (CCR) model assesses the efficiencies of DMUs under the constant returns to scale assumption. Banker et al. [2] formulated the Banker, Charenes and Cooper (BCC) model as an extension of the CCR model to consider variable returns to scale. The standard DEA models, has recently made a substantial contribution in analyzing various environments and numerous applications. A common assumption in DEA literature is real-valued input/outputs. However, it seems questionable to assume that the data and observation are crisp and real-valued in a real environments. However, making appropriate decisions in today's complex world has become

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a challenge for managers and organizations. In the real world, assigning crisp numerical values to evaluate efficiency and productivity is not always possible due to inherent ambiguity. One of the possible answers lies in the field of complex-numbers theory. One of the major contribution of these numbers includes electric circuit. A simple electric circuit consists of conductors and semi-conductor and semi-conductor elements, which is the path of the flow of current. If the components of the circuit are electric, it is called an electric circuit, and if some parts are electric and some are electronic, it is an electronic circuit. Generally, a circuit starts at one point and ends at another point. In other words, a circuit must form a loop. Linear electric circuits are a type of these circuits consist of a current or voltage source and linear elements such as resistance. Each electrical circuit consists of the components such as a power source (battery or generator), Circuit connecting wires or strips, which are made of a material that conducts electricity well, such as copper and circuit components such as capacitor, resistor, inductor, diode, and transistor and so on.

Electrical circuits are generally divided into two categories: series circuits and parallel circuits. Series circuits are one of the most regular circuits that have several different parts and commonly used in household appliances. Series circuits may have problems because they are regularly connected to each other. For example, if a part in this circuit has a problem, all the related parts connected to that part will also fail. On the other hand, parallel circuits supply the energy source of electric current. In this circuit, all parts use the same amount of voltage in the same way and size. But the point is, if one of the equipment has a problem, it does not affect the other equipment. Consequently, assigning crisp numerical values to evaluate efficiency and productivity is not always possible due to inherent ambiguity. Since, the need to deal with complex-valued data in DEA naturally occurs when ones evaluating circuits performance or efficiency especially in electronic engineering. Surveying DEA literature, non-crisp data has been studied from different point of views.

Sobhan Sarkar et.al [3] proposed a super efficiency DEA approach in presence of Z-numbers. Employing the proposed approach, the authors aimed to selecting the sustainable suppliers. The results of this study revealed that the proposed framework can be effectively applied as a high-performing decision-support tool for decision-making under imprecision in the domain of sustainable supplier Selection. Manuel Arana-Jiménez et.al [4] argued the circumstances when their inputs and outputs are described under uncertainty of the type of integer interval data. The authors highlighted the differences of the existing integer approaches and the proposed uncertain-integer data scenarios via extending the integer interval arithmetic and LU-partial orders to fuzzy integer intervals. Equipped with novel computation, two crisp linear optimization models were implemented in order to estimate the efficiency status of each DMU. Over the last two decades, many researchers have proposed DEA-based models with stochastic data.

Amirteimoori et.al [5] in a textbook by the name of "Stochastic benchmarking: Theory and applications" provides a comprehensive reference on quantitative performance evaluation. This book also extends important subjects in production theory into stochastic environment including simple examples and real-life applications As regards Fuzzy DEA, there have been also many developments, for example, Manuel Arana-Jiménez et.al [6] deals with the problem of efficiency assessment when the input and output data are given as fuzzy sets. In particular, a fuzzy extension of the measure of inefficiency proportions, a well-known slacks-based additive inefficiency measure, is considered.

Omrani et.al [7] developed a robust credibility DEA model with fuzzy data set. The authors employed a robust optimization approach to consider uncertainty in constructing fuzzy sets. Moreover, perturbation level is considered as exact and fuzzy values. According to the results, as perturbation degree increases, DMUs get normalized lower efficiencies and vise-versa.

Hosseinzadeh Lotfi et.al [8] in chapter four the book by the title of "Uncertainty in DEA: Fuzzy and belief degree-based uncertainties" discusses the topics of ranking, sensitivity, and stability analysis in fuzzy DEA. Further, the authors examined the sensitivity of the models against the variation on the exogenous inputs and deliver the final outputs of the DMU. Edalatpanah [9] utilized DEA approach for hospital performance measurement in presence of neutrosophic data. In real world, some data are uncertain, indeterminate and inconsistent and vague. The author deals with these data set and proposed a DEA-based model for resolving the problem.

Maghbouli et.al [10] developed he axiomatic foundation for DEA models that assumes subsets of input and output variables to be complex-valued data. It is essential that all real and integer valued are subsets of complex numbers. To best of our knowledge, there is not any especial formulation replying the complex-valued questions especially evaluating performance analysis in electronic circuits. The need to deal with complex-valued data in DEA persists because real- valued data set are contained in complex data set. On the other hand, complex valued numbers are frequently used in engineering problems especially electronic engineering. Therefore, modeling these numbers has attracted renewed interest in the recent DEA literature. The question is how we should formulate the DEA models for these quantities. The contribution of this paper examines complex-valued data in order to gain a deeper understanding of a system in presence of imprecise data. Also, it fills the gap by developing the standard DEA models for complex-valued measures. The rest of the paper is organized as follows. The next section provides a brief review of standard DEA models followed by complex valued data in Section 3. An empirical example is represented in Section 4. Finally the Conclusion is presented to summarize the results and suggests avenues for future research.

2 | Preliminaries

Assume a set of DMUs indexed by J, for all $j \in J = \{1,\dots,n\}$, DMU_j uses m inputs $x_{ij}(i=1,\dots,m, j=1,\dots,n)$ to produce s outputs $y_{rj}(r=1,\dots,s, j=1,\dots,n)$. Also, for each $j \in J = \{1,\dots,n\}$, it is assumed that all input and output vectors require semi-positive and real-valued information. Equipped with the following basic axioms for the Constant Return To Scale (CRS) production set T_C.

I. Inclusion, for all $j \in J = \{1,\dots,n\}$, $(x_j, y_j) \in T_C$.

II. Convexity, T_C is a convex set.

III. CRS, for all $(x, y) \in T_C$ and $\alpha \in \mathbb{R}^{+}: (\alpha x, \alpha y) \in T_C$.

IV. Free Disposability, for all $(x,y) \in T_C$, $(u,v) \in R_{+}^{m+s}$ and $y \geq v: (x+u, y-v) \in T_C$.

V. Minimum extrapolation: T_C is the intersection of all sets that satisfies in axioms I-V.

The associated production possibility set T_C is constructed by Charnes et.al [1] as follows:

$$T_C = \{(x,y) \,|\, x \geq \sum_{j=1}^{n} \lambda_j x_j, y \leq \sum_{j=1}^{n} \lambda_j y_j, \lambda_j \geq 0, \text{ for all } j\}. \quad \textbf{(1)}$$

For evaluating the relative efficiency of under evaluated DMU ($DMU_o(o \in J = \{1,\dots,n\})$) on the basis of above axioms, the input-oriented Charnes, Cooper and Rhodes (CCR) multiplier model [1] is employed as follows:

$$\begin{aligned}
&\theta_o^* = \max \frac{\sum_{r=1}^{s} \hat{u}_r y_{ro}}{\sum_{i=1}^{m} \hat{v}_i x_{io}} \\
&\text{s.t.} \\
&\frac{\sum_{r=1}^{s} \hat{u}_r y_{rj}}{\sum_{i=1}^{m} \hat{v}_i x_{ij}} \leq 1 \ , \ j=1,\dots,n \\
&\hat{u}_r, \hat{v}_i \geq \varepsilon \ , \ r=1,\dots,s, i=1,\dots,m.
\end{aligned} \quad \textbf{(1)}$$

In the above CCR multiplier *Model (1)*, $\varepsilon > 0$ is a non-Archimedean and sufficiently small number. Also, $u_r(r=1,\dots,s)$ and $v_i(i=1,\dots,m)$ are the nonnegative related weights. The above nonlinear programming can be converted to linear programming by Charnes-Cooper transformation [11]:

$$\begin{aligned}
&\theta_o^* = \max \sum_{r=1}^{s} u_r y_{ro} \\
&\text{s.t.} \\
&\sum_{i=1}^{m} v_i x_{io} = 1 \\
&\sum_{r=1}^{s} u_r y_{rj} - \sum_{i=1}^{m} v_i x_{ij} \le 0 \;, \; j=1,\dots,n \\
&u_r, v_i \ge \varepsilon \;,\; r=1,\dots,s, i=1,\dots,m.
\end{aligned} \tag{2}$$

The optimal objective function value of linear *Model (2)* gives the efficiency measure for DMU_o. The Efficiency ratio in *Model (2)* ranges between zero and one, with DMU_o being considered relatively efficient if it receives a score of one. Otherwise it is inefficient. Tone [12] has defined the Slack-Based Measure (SBM) of efficiency of $DMU_o(o \in J = \{1,\dots,n\})$ as the optimal objective function value δ_o that can be obtained from the following program:

$$\begin{aligned}
&\text{Min} \quad \delta_o = t - \frac{1}{m}\sum_{i=1}^{m} \frac{s_i^-}{x_{io}} \\
&\text{s.t.} \\
&t + \frac{1}{s}\sum_{r=1}^{s} \frac{s_r^+}{y_{ro}} = 1 \\
&\sum_{j=1}^{n} \lambda_j x_{ij} + s_i^- = t x_{io} \quad i=1,\dots,m \\
&\sum_{j=1}^{n} \lambda_j y_{rj} - s_r^+ = t y_{ro} \quad r=1,\dots,s \\
&s_i^-,\; s_r^+, \lambda_j \ge 0 \quad , j=1,\dots,n, r=1,\dots s, i=1,\dots,m.
\end{aligned} \tag{3}$$

In the above *Model (3)*, δ_o is the L_1 - distance of DMU_o from to the production frontier spanned by $DMU_j(j \in J)$. DMU_o Is called efficient in evaluation with SBM *Model (3)* if and only if it is efficient in all inputs and outputs, and this means that there is no input excesses and output shortfalls, i.e. $s_i^- = s_r^+ = 0,$ for all i, r.

3 | Complex-Valued Data Envelopment Analysis

As far as we know, DEA has been a challenging subject of evaluation. All standard DEA models including CCR and the extended BCC model assume that all inputs and outputs can be taken real–valued quantities. One of the critical improvements in the field of performance analysis is related to circumstances where the inputs and outputs is not characterized as real-valued quantities. However, there are different studies in the literature address non-crisp data. For example, integer data set Kazemi Matin [13] ,Omrani et.al [7], Manuel Arena Jimenez et.al [4], Moghaddas et.al [14], fuzzy data set Storto [15], Phi-Hung Nguyen et.al [16], z-numbers Sobhan Sarkar et.al [3], Azadeh and Kokabi [17], Salman Nazari-Shirkouhi et.al [18], Stochastic DEA Amirteimoori et.al [5] and so on. But the use of complex-numbers in performance analysis considered as an extension of the computation with numbers in classic DEA modeling.

Complex numbers are commonly used in many varieties of engineering such as electronics, electromagnetism, computer science, mechanical, civil and control systems and so on. One of the application of complex-valued numbers are often built on analyzing the electrical circuits. Electronics circuit is mainly based on voltage and current. These two elements are put together as single complex numbers. Z=V+iI is the complex representation of a circuit having both current and voltage where V is the real axis part and I is the imaginary axis part so that we can able to see the comparison of both V and I by representing as a complex number in electronics. Sometimes in RC circuits or RLC circuits, for combining two elements, for example in the case of resistor and inductor it is written as $R+jX_L$ and in the case of resistor and capacitor, complex number representation would be $R+jX_C$ where $X_L = jwL$ and $X_C = \frac{1}{jwC}$.

To deal with performance analysis in presence of complex-valued, it is necessary to introduce a new operation for transforming the complex-valued into real data. Again, assume that a $DMU_j(j=1,\dots,n)$ yields $\tilde{y}_{rj}(r=1,\dots,s,\ j=1,\dots,n)$ amount of outputs r by consuming $\tilde{x}_{ij}(i=1,\dots,m,\ j=1,\dots,n)$ amount of input i .Here, $\tilde{y}_{rj}(r=1,\dots,s,\ j=1,\dots,n)$ and $\tilde{x}_{ij}(i=1,\dots,m,\ j=1,\dots,n)$ are complex-numbers. These numbers are considered as $\tilde{x}_{ij} = \tilde{x}_{1ij} + i\tilde{x}_{2ij}$ and $\tilde{y}_{rj} = \tilde{y}_{1rj} + i\tilde{y}_{2rj}$, whereas, $\tilde{x}_{1ij}$ and $\tilde{y}_{1rj}$ represent the real part of inputs and output vectors, respectively. Also, $\tilde{x}_{2ij}$ and $\tilde{y}_{2rj}$ denotes the imaginary part of input and outputs. However, it is not straightforward to solve a DEA model in the presence of complex-numbers. Therefore, employing the approach below, we have converted the complex-numbers into real–valued data.

Definition 1. Consider $z = x + iy$ as a complex-valued numbers. Then, the ranking function $\varepsilon(z)$ can be defined with *Eq. (4)*:

$$\varepsilon(z) = \frac{|z|^2 + 2\,\mathrm{Re}(z) + \mathrm{Im}(z)}{12}. \qquad \textbf{(4)}$$

In the definition above, $|z|$ keeps the magnitude or norm of a complex-number, whereas the elements $\mathrm{Re}(z)$ and $\mathrm{Im}(z)$ represent the reality and imaginary of the complex number, respectively. Motivating with this real quantity, consider the following axioms analogous to (I) to (V) in Section2, in presence of complex-valued inputs and outputs with employing the ranking function of *Eq. (4):*

I. Inclusion, for all $j \in J = \{1,\dots,n\}$, $(\tilde{x}_j, \tilde{y}_j) \in T$.
II. Convexity, T is a convex set.
III. CRS, for all $(\tilde{x}\ ,\tilde{y}\ ) \in T$ and $\alpha \in \square^+$: $(\alpha\tilde{x}, \alpha\tilde{y}\ ) \in T$.
IV. Free Disposability, for all $(\tilde{x},\tilde{y}) \in T$, $(u,v) \in \square$ and $\varepsilon(y) \ge \varepsilon(v)$: $(\tilde{x}+u, \tilde{y}-v) \in T$.
V. Minimum extrapolation: T is the intersection of all sets that satisfies in axioms I-V.

The reference technology in presence of complex-valued inputs and outputs can be stated as:

$$T = \left\{ (x,y) \in \square \;\middle|\; \varepsilon(x) \ge \sum_{j=1}^{n} \lambda_j \varepsilon(x_j), \varepsilon(y) \le \sum_{j=1}^{n} \lambda_j \varepsilon(y_j), \lambda_j \ge 0, j \in J \right\}. \qquad \textbf{(5)}$$

Representing the real-valued matrix by employing the *Eq. (4)*, both radial and non-radial standard DEA models are satisfied. It is worth to note that the common efficiency measures all assume continuous real-valued data. To measure the efficiency measure potential in complex-valued variables, the Farrell efficiency measure is modified as follows:

$$\mathrm{Eff} = \min\left\{ \theta \in R \,\middle|\, \exists (\tilde{x},\tilde{y}) \in T; \theta\varepsilon(x_o) \ge \varepsilon(\tilde{x}), \varepsilon(y_o) \le \varepsilon(\tilde{y}) \right\}.$$

Hence, examining the complex numbers with a real –valued ranking function as mentioned in *Eq. (4)*, the efficiency measure gauges radial distance to the monotonic hull of the production possibility set. This arranging preserve the usual interpretation of the Farrell measure as a downward scaling potential in inputs as the given output levels. Also guarantees that the units with the score one are efficient. Consequently, the modified matrix are used in standard CCR input oriented in order to measure the efficiency score in presence of complex-valued numbers. It should be noted that applying the *Eq. (4)* of each element dare to need a complex calculation. However, this modification is time consuming especially in large-scale problems, but coincides the CRS axioms. Hence, we restrict attention only to data set and applying the CCR measure directly to modified data set. The modified input efficiency scores relative to the modified reference technology can be computed by following linear programming, which is computed by standard algorithms and solver software.

$$
\begin{aligned}
& \text{Eff} = \text{Min} \quad \theta \\
& \text{s.t.} \\
& \sum_{j=1}^{n} \lambda_j \varepsilon(\tilde{x}_{ij}) \le \theta \varepsilon(\tilde{x}_{io}), \quad i = 1,\dots,m \\
& \sum_{j=1}^{n} \lambda_j \varepsilon(\tilde{y}_{rj}) \ge \varepsilon(\tilde{y}_{ro}), \quad r = 1,\dots,s \\
& \lambda_j \ge 0 \quad j = 1,\dots,n.
\end{aligned} \tag{6}
$$

The *Model (6)* above directly deal with real-valued quantities, $\varepsilon(\tilde{x}_{ij})$ and $\varepsilon(\tilde{y}_{rj})$. Also, the intensity weight $\lambda_j (j \in J)$ need to be nonnegative. Note that the objective function is a real number and ranges between zero and one, i.e. $\theta^* \in [0,1]$. It should be noted that a DMU have no inefficiency if in optimality it achieves the score of one. Otherwise, it is called inefficient.

Theorem 1. for any complex-valued data, $(\tilde{x}_o, \tilde{y}_o)$ the optimal θ^* is equal to the modified Farrell efficiency measure Eff defined with respect to the reference *Eq. (5)*.

Proof: the proof is straightforward regarding to CCR input-oriented envelopment model.

Based on *Eq. (4)*, the aim of *Model (6)* is to obtain the efficiency measure of systems with complex-valued data set under the CRS conditions. The framework involves transforming the complex-data into real-valued quantities, then measuring the efficiency through standard DEA models. Through the incorporating slacks (i.e. input saving and output surpluses) into the SBM *Model(3)* along with employing *Eq. (4)* for transforming the complex data into real-valued quantities, the modified SBM model in presence of complex-valued inputs and outputs are represented as follows:

$$
\begin{aligned}
& \text{Min} \ \delta_o = t - \sum_{i=1}^{m} \frac{s_i^-}{\varepsilon(\tilde{x}_{io})} \\
& \text{s.t.} \\
& t + \frac{1}{s} \sum_{r=1}^{s} \frac{s_r^+}{\varepsilon(\tilde{y}_{ro})} = 1 \\
& \sum_{j=1}^{n} \lambda_j \varepsilon(\tilde{x}_{ij}) + s_i^- = t\varepsilon(\tilde{x}_{io}) \quad i = 1,\dots,m \\
& \sum_{j=1}^{n} \lambda_j \varepsilon(\tilde{y}_{rj}) - s_r^+ = t\varepsilon(\tilde{y}_{ro}) \quad r = 1,\dots,s \\
& s_i^-,\ s_r^+, \lambda_j \ge 0,\ j = 1,\dots,n, r = 1,\dots s, i = 1,\dots,m.
\end{aligned} \tag{7}
$$

The real-valued $\varepsilon(\tilde{x}_{ij})$ and $\varepsilon(\tilde{y}_{rj})$ can assist the manager in deciding the efficient and inefficient units with complex-valued quantities. All constraints in *Model (7)* are the usual constraints of SBM *Model (3)*, but *Model (7)* employs the transformed values of complex inputs and outputs. It is sufficient to guarantee a unit in *Model (7)* may be efficient in some inputs or some outputs but not overall since it could be inefficient in aggregate sense. It is worth to note that, the DMUs with complex valued cannot rank based on these scores. Therefore, we have used the ranking index below, denoting by RI_j, as given in *Eq. (8)*:

$$RI_j = \frac{1-\frac{1}{2m}\left[\sum_{i=1}^{2m}\frac{s_i^{-*}+(1-\delta_0^*)\varepsilon(\tilde{x}_{io})}{\varepsilon(\tilde{x}_{io})}\right]}{1+\frac{1}{2s}\left[\sum_{r=1}^{2s}\frac{s_r^{+*}}{\varepsilon(\tilde{y}_{ro})}\right]}. \tag{8}$$

Based on this ranking index, we can rank the DMUs in its descending order, i.e., larger value of RI_j is better for $DMU_j (j \in J)$. The variables s_i^{-*} and s_r^{+*} are the optimal values of slack defined in *Model (7)*, furthermore, δ_o^* is the optimal value of *Model (7)*.

4 | Numerical Experiment

In this section, a case study of 15 complex inputs and outputs from a simple circuit including capacitor, resistance and rewind (CRR circuit) is used to reveal the validity and usefulness of the proposed models. The circuit is arranged as *Fig. (1)* shows:

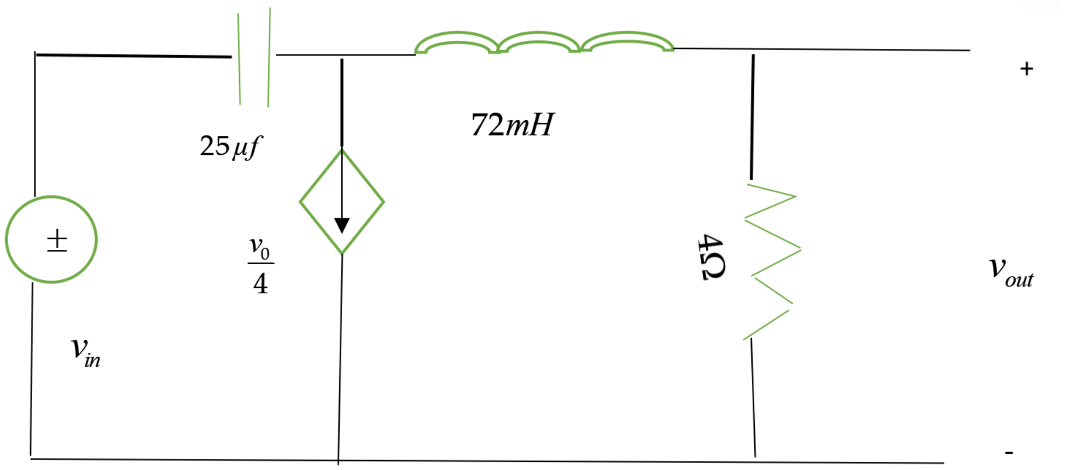

Fig. 1. A schematic diagram of CRR circuit.

The input and output voltage are described in *Table 1*. Each input and output are represented in its exponential format. Notably, each iteration is considered as a Decision Making Unit (DMU) along with its input and output in related system depicted in *Fig. (1)*. *Table 1* shows the data set in both representation of complex numbers employing the Euler formulae.

Table 1. The complex data set.

DMU	Input Voltage	Output Voltage
1	$r=20,\theta=0$ $z=20+0i$	$r=8.944,\theta=63.43,$ $z=3.99+5.036i$
2	$r=40,\theta=0$ $z=40+0i$	$r=17.88,\theta=63.43$ $z=14.755+10.068i$
3	$r=60,\theta=0$ $z=60+0i$	$r=26.83,\theta=63.43$ $z=22.171+15.108i$
4	$r=80,\theta=0$ $z=80+0i$	$r=35.77,\theta=63.43$ $z=29.560+20.142i$
5	$r=100,\theta=0$ $z=100+0i$	$r=44.72,\theta=63.43$ $z=36.956+25.182i$
6	$r=20,\theta=20$ $z=8.161+25.259i$	$r=8.938,\theta=83.44$ $z=-1.668+8.871i$
7	$r=20,\theta=40$ $z=-13.338+14.902i$	$r=8.72,\theta=-76.215$ $z=5.970-6.356i$
8	$r=20,\theta=60$ $z=-19.048-6.096i$	$r=8.94,\theta=-56.55$ $z=-8.940-0.012i$
9	$r=20,\theta=80$ $z=-2.207-19.877i$	$r=8.96,\theta=-36.77$ $z=-5.363+7.177i$
10	$r=20,\theta=100$ $z=17.246-10.127i$	$r=8.94,\theta=-16.56$ $z=5.886+6.728i$
11	$r=40,\theta=20$ $z=16.323+36.518i$	$r=17.87,\theta=83.43$ $z=-3.160-15.676i$
12	$r=40,\theta=40$ $z=-26.677+29.804i$	$r=17.88,\theta=-76.564$ $z=-7.045-16.433i$
13	$r=40,\theta=60$ $z=-38.096-12.192i$	$r=17.88,\theta=-56.55$ $z=17.880-0.024i$
14	$r=40,\theta=80$ $z=-4.415-39.755i$	$r=17.88,\theta=-36.538$ $z=7.122+16.400i$
15	$r=40,\theta=100$ $z=34.493-20.254i$	$r=17.887,\theta=-16.5414$ $z=-12.214+13.241i$

The results of implementing of *Model (6)* and *Model (7)* are shown in *Table 2*. Utilizing the *Eq. (4)*, $\varepsilon(z)$ the data set are transformed into real-valued data. *Table 2* depicts the real-valued data set.

Table 2. Real-valued Data Set.

DMU	Input Voltage	Output Voltage
1	36.67	4.52
2	140	29.89
3	310	64.94
4	546.67	113.23
5	850	174.91
6	62.18	7.25
7	32.35	6.8
8	29.65	5.17
9	31.31	6.39
10	35.36	8.2
11	139.1	19.48
12	131.37	24.1
13	125.96	29.62
14	129.28	29.19
15	137.39	26.11

Equipped with the real values, the efficiency scores for the transformed data set are presented in *Table (W)*. As we can see in the *Table 3*, all DMUs are classified as either efficient or inefficient. As *Table 3* depicts, there is only one efficient unit in this evaluation. DMU #13 is the only efficient unit with the efficiency score of unity. The fourth and fifth columns of *Table 3* represents the input excess and output shortfall in evaluation with *Model 7*. DMU #15 is the only unit with the input excess quantity of 0.45, whilst the other units has recorded no excess in their inputs. The output shortfalls are recorded zero for the only efficient unit, #13. The SBM *Model 7* suggests generally the same efficiency level as *Model 6* records.

Table 3. The efficiency score of models (6) and (7).

DMU	Eff^* (model 6)	δ_o^* (model 7)	s_i^{-*}	s_r^{+*}	RI_j	Ranking
1	0.52	0.52	0	2.15	0.614	14
2	0.91	0.91	0	2.75	0.914	4
3	0.89	0.89	0	7.09	0.897	6
4	0.88	0.88	0	13.5	0.888	7
5	0.88	0.88	0	21.85	0.885	8
6	0.5	0.5	0	3.66	0.599	15
7	0.89	0.89	0	0.72	0.898	5
8	0.74	0.74	0	1.34	0.771	12
9	0.87	0.87	0	0.84	0.878	9
10	0.99	0.99	0	0.11	0.989	2
11	0.6	0.6	0	7.88	0.666	13
12	0.78	0.78	0	5.3	0.802	11
13	1	1	0	0	1	1
14	0.98	0.98	0	0.5	0.982	3
15	0.81	0.81	0.45	4.92	0.826	10

The sixth column of *Table 3* under the heading of "RI_j" signifies the index of rankings for DMUs employing *Eq. 8*. The last column of *Table 3* shows the ranking of units. As the results underlines, unit13 has the first rank, the second and third ranks assigned to units #10 and #14. *Fig. 2* demonstrated the ranking index of fifteen DMUs which are strongly correlated with the original ranking.

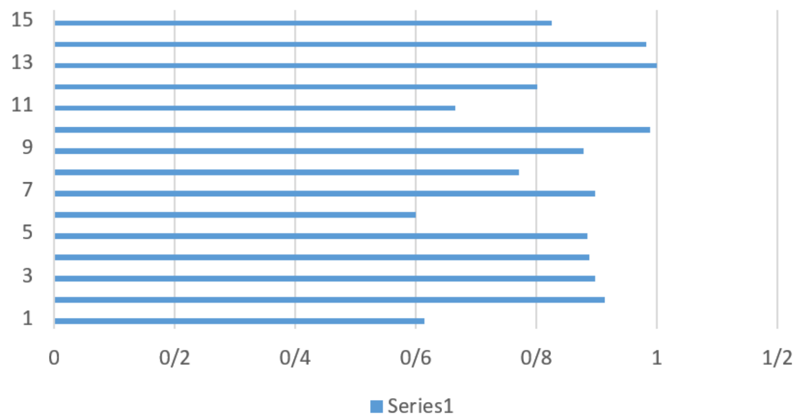

Fig. 2. The values of the ranking index.

To support this statement, the difference between the efficiency measure δ_o^* obtained from *Model (7)* and the data obtained from RI_j is insignificant. Therefore, we can infer that the proposed model is robust and less sensitive. This suggests that the index can enhance the discriminatory power of the model considerably. One more point to be noted tis the approximation of each complex element through the real valued quantity. However, this approximation could suggest an avenue for handling complex data but it seems that we need further investigation in this field.

5|Conclusion

Treating imprecise data attracts the interest of DEA researchers since the last decade. In the realm of efficiency and productivity analysis various approaches have been developed within DEA research to address the concept of imprecise data. This study examined the extension of standard DEA models to deal with complex-valued inputs and outputs. Implementing a measure of approximation of complex-value data with a real magnitude, the efficiency score was evaluated. Also, a ranking order has been extracted. Finally, the application of the proposed approach was examined in a real-case of an electric circuits including fifteen complex inputs and outputs. The proposed approach is appropriate in situations where some inputs or outputs do not have a real-valued quantitative value, and the proposed approach has produced promising results from computing efficiency and performance aspects. Although the study faces some barriers. The proposed models were investigated under CRS, but they can be extended under Variable Return To Scale (VRS) assumption, so we plan to extend this model to VRS. Moreover, the arithmetic operation for transformation to a real-valued quantity here may lead to bias in other type of DEA structures such as network DEA. This subject can be then investigated in future studies. What's more, the application of the demonstrated approach can also be considered in different areas such banks, supplier selection, and so on. As for future research, the problems can be studied.

Author Contributions

Conceptualization, M. Maghbouli. A. PourhabibYekta. and M. Zorriehhabib; methodology, A. PourhabibYekta; software, M. Maghbouli; validation, M. Zorriehhabib; formal analysis, M. Maghbouli. A. PourhabibYekta. and M. Zorriehhabib; investigation, M. Maghbouli and M. Z orriehhabib; resources, M. Maghbouli. A. PourhabibYekta. and M. Zorriehhabib; data curation, M. Zorriehhabib; writing—original draft preparation. M. Maghbouli; writing—review and editing, A. PourhabibYekta; visualization, M. Zorriehhabib; All authors have read and agreed to the published version of the manuscript.

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Data Availability Statement

The Data used in the manuscript are experimental data and are provided through an empirical experiment in the laboratory.

Conflicts of Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper. The funders had no role in the design of the study; in the collection, analyses, or interpretation of data; in the writing of the manuscript, or in the decision to publish the results.

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